08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie Quantum Field Theory I

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Homework set 3

Due November 14, 2022 by start of discussion session.
Please note how long it took you to solve each problem.

3-1, 40 pts. The Lorentz group. Read section 3.1 of Peskin and Schroeder (and refer to Lecture 2 notes as needed). The generators of the Lorentz group $J^{\mu\nu}$ obey the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right) \ . \tag{1}$$

We can furnish specific representations by assigning $J^{\mu\nu}$ to different mathematical objects that obey the above commutation relations.

A, 6 pts. How many generators are in the Lorentz group (for 3+1 spacetime dimensions)? What Lorentz transformations do they correspond to?

B, 10 pts. Demonstrate that $J^{\mu\nu} = L^{\mu\nu}$, with $L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$, is a faithful representation by verifying that $L^{\mu\nu}$ obeys the commutation relations.

C, 10 pts. Demonstrate that $J^{\mu\nu} = \mathcal{J}^{\mu\nu}$, with $(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right)$ is also a faithful representation. This is known as the vector representation, since the corresponding 4×4 matrices act on 4-vectors.

D, 14 pts. For the vector representation, evaluate the explicit form of the transformation matrix $U = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)$ for the following special cases:

i, 7 pts. $\omega_{13} = -\omega_{31} = \theta$, $\omega_{\mu\nu} = 0$ otherwise. Which axis of rotation does this correspond to?

ii, 7 pts. $\omega_{03} = -\omega_{30} = \beta$, $\omega_{\mu\nu} = 0$ otherwise. What boost does this correspond to?

3-2, 12 pts. Spinor space. Use the explicit form of the Pauli matrices to compute the eigenvalues of $p\cdot\sigma$ and $p\cdot\bar{\sigma}$, where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$. Show that for an on-shell particle ($p^2 = m^2$, $p^0 > 0$) that the eigenvalues are always positive. Obtain the explicit form of the spinor $u_s(p)$ for a particle moving in the $+\hat{x}$ direction.

3-3, 12 pts. Gordon identity (problem 3.2 of Peskin and Schroeder). Derive the Gordon identity,

$$\bar{u}_r(p')\gamma^\mu u_s(p) = \bar{u}_r(p')\left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u_s(p) \tag{2}$$

for $q = p' - p$. You can use the constraint $(\not{p} - m)u_s(p) = 0$ on the spinor from the Dirac equation.

3-4, 12 pts. Lorentz transformations of bilinear spinor contractions. Given that $\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu$ for the Dirac matrices γ^μ and defining $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, show that under a Lorentz transformation

A, 6 pts. $\bar{\psi}\gamma_5\psi \to \det(\Lambda)\bar{\psi}\gamma_5\psi \quad ,$

B, 6 pts. $\bar{\psi}\gamma^\mu\gamma_5\psi \to \det(\Lambda)\Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\gamma_5\psi \quad .$

[Aside: The first fermion bilinear is a pseudoscalar contraction, and the second fermion bilinear is a pseudovector contraction. In general, $\det(\Lambda) = +1$ for continuous Lorentz transformations, but we can also have $\det(\Lambda) = -1$ if we perform a discrete Lorentz transformation such as a spatial reflection (known as parity).]

3-5, 24 pts. Practice with Dirac algebra (part 1). Evaluate the following products of γ matrices with contracted indices. (Hint: See equation 5.9 of Peskin and Schroeder for some solutions.)

A, 4 pts. $\gamma^\mu\gamma_\mu$

B, 4 pts. $\gamma^\mu\gamma^\nu\gamma_\mu$

C, 4 pts. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu$

D, 4 pts. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu$

E, 4 pts. $(\gamma_5)^2$

F, 4 pts. Define

$$P_L = \frac{1_{4\times4} - \gamma_5}{2} \tag{3}$$

$$P_R = \frac{1_{4\times4} + \gamma_5}{2} \ . \tag{4}$$

Show that $(P_L)^2 = P_L$, $(P_R)^2 = P_R$, and $P_L P_R = 0$.