

Lecture 5.

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Quantum Field Theory I.

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Last time: Noether's theorem, time-dependence

Today: Antiparticles + time dependence.

~~Scalar~~ Scalar propagator and KG Green's fcn.

Last time, we discussed the commutator $[\phi(x), \phi(y)]$:

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} (e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}) \\ &= D(x-y) - D(y-x) \end{aligned}$$

We also saw that the commutator vanished for spacelike intervals and was nonzero for timelike intervals, which was a necessary condition for causality.

It is more instructive to consider the question of causality when we can distinguish particle and antiparticle. We can do this easily when a charge distinguishes the two, ~~as~~ as can be seen for a complex scalar field.

With complex scalar fields, the negative frequency modes are associated with antiparticles, and we must introduce a new set of creation and annihilation operators:

$b_{\vec{p}}^+$ and $b_{\vec{p}}$.

Remark: $\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip \cdot x} a_{\vec{p}} \Big|_{p^0=E_{\vec{p}}}$

is wrong + fails causality (although commutation relations do hold for $\phi + \phi^*$).

So, we write

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$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (e^{-ip \cdot x} a_{\vec{p}} + e^{ip \cdot x} b_{\vec{p}}^\dagger) \quad ; p^0 = E_{\vec{p}}$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (e^{-ip \cdot x} b_{\vec{p}} + e^{ip \cdot x} a_{\vec{p}}^\dagger)$$

$$\pi(x) = \dot{\phi}^*(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (-iE_p) (e^{-ip \cdot x} b_{\vec{p}} - e^{ip \cdot x} a_{\vec{p}}^\dagger)$$

$$\pi^*(x) = \dot{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (-iE_p) (e^{-ip \cdot x} a_{\vec{p}} - e^{ip \cdot x} b_{\vec{p}}^\dagger)$$

To ensure the canonical commutation relations, we treat a & b as their own ladder operators:

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = \delta^{(3)}(\vec{p} - \vec{q}) = [b_{\vec{p}}, b_{\vec{q}}^\dagger]$$

All other commutators vanish: $[a_{\vec{p}}, a_{\vec{q}}] = [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0$

$$[b_{\vec{p}}, b_{\vec{q}}] = [b_{\vec{p}}^\dagger, b_{\vec{q}}^\dagger] = 0$$

Mixed commutators vanish!
also

Can derive free Hamiltonian and momentum operator:

$$H_{\text{free}} = \int \frac{d^3p}{(2\pi)^3} E_p (a_{\vec{p}}^\dagger a_{\vec{p}} + b_{\vec{p}}^\dagger b_{\vec{p}} + \text{zero pt. energy})$$

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p} (a_{\vec{p}}^\dagger a_{\vec{p}} + b_{\vec{p}}^\dagger b_{\vec{p}})$$

Interpretation:

Recall that $\phi(x)$ and $\phi^*(x)$ transform oppositely under a U(1) phase transformation: if $\phi(x) \rightarrow e^{i\alpha} \phi(x)$, then $\phi^*(x) \rightarrow e^{-i\alpha} \phi^*(x)$. Correspondingly, we ~~can~~ ^{must} interpret $b_{\vec{p}}^\dagger$ and $a_{\vec{p}}$ consistently with regards to charge flow.

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Since $\phi(x)|0\rangle$ creates a particle at position $\vec{x}$, we exercise the $b_p^\dagger$ operator and further specify that $\phi(x)|0\rangle$ creates a + charge particle at position $\vec{x}$ via $b_p^\dagger$ or annihilates a - charge particle at position $\vec{x}$ via a_p . Similarly, $\phi^*(\vec{x})|0\rangle$ creates a - charge particle at position $\vec{x}$ with $a_p^\dagger$ or annihilates a + charge particle at position $\vec{x}$ with b_p .

So, $b_p^\dagger + b_p$ create + annihilate + particles and $a_p^\dagger + a_p$ create + annihilate the - antiparticles.

Then, if we consider now $[\phi(x), \phi^\dagger(y)] = D(x-y) - D(y-x)$, the first term $\phi(x)\phi^\dagger(y) = D(x-y)$ is the creation of a - charge at y that travels to x and is annihilated; the second term $\phi^\dagger(y)\phi(x) = D(y-x)$ is the creation of a + charge at x that travels to y and is annihilated. To be self-consistent, the particle + antiparticle must both exist, have equal masses, and have opposite charges. The real scalar field has no charge, so the antiparticles + particles are equivalent.

Klein-Gordon propagator

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \\ = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right) \end{aligned}$$

Change to d^4p integral by considering integrand as ~~result~~ result of dp^0 integral.

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$$= \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip \cdot (x-y)} \Big|_{p^0=E_p} + \frac{1}{2E_p} e^{-ip \cdot (x-y)} \Big|_{p^0=-E_p} \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \int \frac{dp^0}{(2\pi i)} \frac{-1}{p^2 - m^2} e^{-ip \cdot (x-y)}$$

Check: $p^2 - m^2$ has poles for

$$p^0 = E_p = \pm \sqrt{|\vec{p}|^2 + m^2} \Rightarrow$$

For $x^0 > y^0$, need $e^{-ip \cdot (x-y)} e^{-ip^0(x^0-y^0)}$ to vanish exponentially fast, so we take p^0 in the lower half-plane. For $x^0 < y^0$, then we close the p^0 integral in the upper half-plane + get 0.

we get 2 residues as above

← to vanish exponentially fast, so we take p^0 in the lower half-plane. For $x^0 < y^0$, then we close the p^0 integral in the upper half-plane + get 0.

Putting these together, we need a Heaviside for. on $(x^0 - y^0)$:

$$D_R(x-y) \equiv \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

This is the retarded Green's function of the Klein-Gordon operator. Vanishes for $x^0 < y^0$.

Check:

$$\begin{aligned} (\partial^2 + m^2) D_R(x-y) &= [\partial^2 \theta(x^0 - y^0)] \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \\ &\quad + \theta(x^0 - y^0) (\partial^2 + m^2) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle \\ &\quad + 2 [\partial_\mu \theta(x^0 - y^0)] (\partial^\mu \langle 0 | [\phi(x), \phi(y)] | 0 \rangle) \end{aligned}$$

$$= [\partial_4 \delta(x^0 - y^0)] \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

$$\phi(x), \phi(y) \rightarrow + 0$$

$$\text{are solutions of KG} \quad + 2 \delta(x^0 - y^0) \langle 0 | [\pi(x), \phi(y)] | 0 \rangle$$

Integrate by parts in first line:

$$= - \delta_4(x^0 - y^0) \langle 0 | [\pi(x), \phi(y)] | 0 \rangle$$

$$+ 2 \delta(x^0 - y^0) \langle 0 | [\pi(x), \phi(y)] | 0 \rangle$$

$$= \delta(x^0 - y^0) (-i \delta^{(3)}(\vec{x} - \vec{y})) = -i \delta^{(4)}(x - y)$$

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We see that $D_K(x-y)$ is a Green's fun. of the Klein-Gordon operator.

Recall: Green's fns. are solutions to differential eqns. that give a δ -fun. response. In particular, they are the building blocks of solving inhomogeneous differential eqns, since we can generally decompose sources into δ -fun. pieces, each solved by its own Green's fun, then superposing the solutions together.

Alternate derivation: Fourier transform to momentum space.

$$D_K(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \tilde{D}_K(p)$$

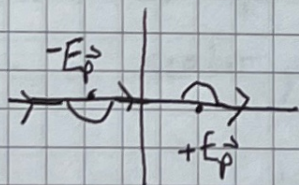
Then $\tilde{D}_K(p)$ is the solution to

$$(-p^2 + m^2) \tilde{D}_K(p) = -i$$

$$\Rightarrow \tilde{D}_K(p) = \frac{i}{p^2 - m^2}$$

$$\Rightarrow D_K(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip \cdot (x-y)}$$

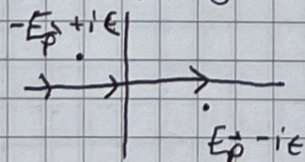
Feynman prescription: Enclose poles with $+i\epsilon$:



Modify denominator to
 $p^2 - m^2 \rightarrow p^2 - m^2 + i\epsilon$

Then, poles are $(p^0 \pm i\epsilon)^2 = |\vec{p}|^2 + m^2 - i\epsilon$
 $\Rightarrow p^0 = E_p - i\epsilon$ and $p^0 = -E_p + i\epsilon$

Pictorially, get contour as



Will then take $\epsilon \rightarrow 0$ at end.

For $x^0 > y^0$, we close the contour below and get

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)}$$

For $x^0 < y^0$, we close the contour above and get

$$D(y-x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{ip \cdot (x-y)}$$

Together, this defines

$$D_F(x-y) = \begin{cases} D(x-y), & x^0 > y^0 \\ D(y-x), & x^0 < y^0 \end{cases}$$

$$= \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle$$

$$\equiv \langle 0 | T \phi(x) \phi(y) | 0 \rangle$$

with T = time-ordering, which rearranges the following operators to put the latest time at the left-most position + earliest all the way to the right. $D_F(x-y)$ = Feynman propagator + reflects the propagation amplitude.

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

This will correspond to a momentum-space Feynman rule for a scalar propagator,

$$\text{---} \overrightarrow{\text{---}} \quad \frac{i}{p^2 - m^2 + i\epsilon} \quad \text{or} \quad \text{---} \overleftarrow{\text{---}}$$

p p

$$(\text{since } D_F(x-y) = D_F(y-x))$$