Lecture 27 Supplement.
Quantum Field Theory I.
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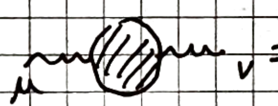
QED renormalization @ 1-loop.

We are now ready to assemble all the calculations, write the renormalization conditions, and define the counterterm Lagrangian & renormalized Lagrangian.

$$\mathcal{L}_{bare} = -\frac{1}{4}\left(F_{b\,\mu\nu}\right)^2 + \bar{\psi}_b\,(i\not\partial - m_0)\,\psi_b - e_0\,\bar{\psi}_b\,\gamma^\mu\,\psi_b\,A_{b\mu}$$

Define $\psi_b \equiv \sqrt{Z_2}\cdot\psi_r$ & $A_{b\mu} = \sqrt{Z_3}\,A_{r\mu}$

These will match the 2-pt. corr. fcn. wavefcn rescaling factors:

 $= \dfrac{i Z_2}{\underbrace{\not p - m}_{\text{single particle}}} + \ldots;$ 

$= \dfrac{-i Z_3\, g_{\mu\nu}}{\underbrace{q^2}_{\text{single particle}}} + \ldots$

$$\mathcal{L}_{bare} = -\frac{1}{4} Z_3 \left(F_r^{\mu\nu}\right)^2 + Z_2\,\bar{\psi}_r\,(i\not\partial - m_0)\,\psi_r - e_0 Z_2\sqrt{Z_3}\,\bar{\psi}_r\gamma^\mu\psi_r A_{r\mu}$$

We define the physical ~~electric~~ QED gauge coupling, $e Z_1 \equiv e_0 Z_2 \sqrt{Z_3}$.
We also have the $Z_2 m_0$ mass, which we split apart into a counterterm & physical mass:

$$Z_2 m_0 \equiv m + \underbrace{\delta m}_{\text{ct.}}$$

Similarly, all Z_i are split into a canonical piece (residue factor 1) and a counterterm:

$$Z_2 \equiv 1 + \delta_2, \quad Z_3 \equiv 1 + \delta_3, \quad Z_1 \equiv 1 + \delta_1$$

So, $$\mathcal{L}_{bare} = -\frac{1}{4}(F_r^{\mu\nu})^2 + \bar{\psi}_r(i\not{\partial}-m)\psi_r - e\bar{\psi}_r\gamma^\mu\psi_r A_{r\mu}$$

$$-\frac{1}{4}\delta_3(F_r^{\mu\nu})^2 + \bar{\psi}_r(i\delta_2\not{\partial}-\delta_m)\psi_r - e\delta_1\bar{\psi}_r\gamma^\mu\psi_r A_{r\mu}$$

The upper line is the renormalized perturbation Lagrangian, and the lower line is the counterterm Lagrangian.

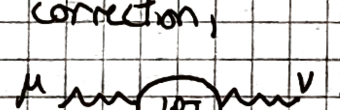
Key point: If we consider the lower line as simply infinite offsets of our physical fields, then the first line is exactly the dynamical Lagrangian we wanted from the beginning.

So, $\mathcal{L}_{bare} = \mathcal{L}_{ren} + \mathcal{L}_{ct}$.

$$\Rightarrow \mathcal{L}_{ren.} = \mathcal{L}_{bare} - \mathcal{L}_{ct}$$

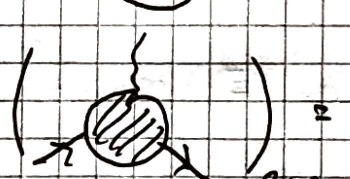
Again, since the cancellation of infinities is mathematically ambiguous, we use renormalization conditions to fix $\delta_1, \delta_2, \delta_3$ & δ_m (which are formally infinite, but we fix the finite remainder after cancellation.)

Given our exact results from 2-pt. corr. fcns & vtx. correction,



$$= i\Pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2 - q^\mu q^\nu)\Pi(q^2)$$

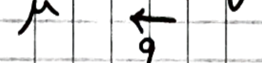
$$= -i\Sigma(\not{p})$$



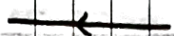
$$= -ie\Gamma^\mu(q)$$

we have 4 renormalization conditions, to be applied when summing over the 1-loop terms in ren. pert. theory & the new Feynman rules induced by the counterterms.

Renormalized pert. theory
Feynman rules:



$$= \frac{-ig_{\mu\nu}}{q^2+i\epsilon}$$



$$= \frac{i}{\not p - m + i\epsilon}$$

$$= -ie\gamma^\mu = ieQ\gamma^\mu$$

"true" m & e here

Counterterm rules

$$= -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta_3$$

$$= i(\not p\,\delta_2 - \delta_m)$$

$$= -ie\gamma^\mu\delta_1$$

The ren. conditions are:

$\Sigma(\not p = m) = 0$ ⇒ physical pole of fermion is at $\not p = m$

$\frac{d}{d\not p}\Sigma(\not p)\Big|_{\not p = m} = 0$ ⇒ residue is 1.

$\Pi(q^2 = 0) = 0$ ⇒ residue is 1.

$-ie\Gamma(q=0) = -ie\gamma^\mu$ ⇒ fix electron coupling to infinitely soft photon / long range force to only γ^μ.

We collect our expressions from before & impose ren. conditions:

$$-i\Sigma_2(p) = -ie^2Q^2\int dx \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)\left(-(d-2)x\not{p} + dm\right)}{\left(x^2p^2 - xp^2 + (1-x)m^2 + x\mu^2\right)^{2-d/2}}$$

We now sum & include counterterms

$$-i\Sigma(\not{p}=m) = -i\Sigma_2 + i(\not{p}\,\delta_2 - \delta_m)\Big|_{\not{p}=m} = 0$$

$$i(m\delta_2 - \delta_m) = \Sigma_2(m) = \frac{e^2Q^2 m}{(4\pi)^{d/2}}\int_0^1 dx \frac{\Gamma(2-d/2)\left(4-2x-\epsilon(1-x)\right)}{\left((1-x)^2m^2 + x\mu^2\right)^{2-d/2}}$$

We also have $\frac{d}{d\not{p}}\Sigma(\not{p})\Big|_{\not{p}=m} = \frac{d}{d\not{p}}\Sigma_2 - \delta_2 = 0$

$$\Rightarrow \delta_2 = \frac{-e^2Q^2}{(4\pi)^{d/2}}\cdot\int dx \frac{\Gamma(2-d/2)}{\left((1-x)^2m^2 + x\mu^2\right)^{2-d/2}}$$

$$\left[(2-\epsilon)x - \frac{\epsilon}{2}\frac{2x(1-x)m^2}{(1-x)^2m^2 + x\mu^2}\left(4-2x-\epsilon(1-x)\right)\right]$$

Apply $\Pi(q^2=0) = \left(\Pi_2(q^2) - \delta_3\right)\Big|_{q^2=0} = 0$

We had $\Pi_2(q^2) = \frac{-e^2Q^2\cdot 8}{(4\pi)^{d/2}}\int dx \frac{\Gamma(2-d/2)\; x(1-x)}{\left(m^2 - x(1-x)q^2\right)^{2-d/2}}$

$$\delta_3 = \Pi_2(0) = \frac{-e^2Q^2\cdot 8}{(4\pi)^{d/2}}\int dx \frac{\Gamma(2-d/2)\; x(1-x)}{(m^2)^{2-d/2}}$$

Finally, our vtx fcn. has the last ren. condition.
From our expression, we isolate the $F_1(q)$ term as the coefficient of $ieQ\gamma^\mu$.

Then $F_1(q^2) = \overset{\text{tree}}{1} + \delta F_1(q^2)$

$$\delta F_1(q^2) = \frac{e^2}{(4\pi)^{d/2}} \int dx\,dy\,dz\;\delta(x+y+z-1)$$

$$\left[\frac{(2-\epsilon)^2}{2} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} + \frac{\Gamma(3-d/2)}{\Delta^{3-d/2}} \left(q^2(2(1-x)(1-y)-\epsilon xy) + m^2(2(1-4z+z^2)-\epsilon(1-z)^2) \right) \right]$$

where $\Delta \equiv (1-z)m^2 + z\mu^2 - xyq^2$.

We set $\delta_1 = -\delta F_1(0)$.

With all of the counterterms set, there are no $\overset{UV}{\text{divergences}}$ anymore when calculating in renormalized pert. theory.

Note that $\delta_1 = \delta_2$, which is a consequence of gauge symmetry and forces $Z_1 = Z_2$ (to all orders!).
Then, $e = \frac{Z_2}{Z_1}\sqrt{Z_3}\, e_0 = \sqrt{Z_3}\, e_0$ is only rescaled by the photon diagram / vacuum polarization.
The vertex + fermion propagator match exactly $\overset{\text{UV divs.}}{}$ in order to ensure there is no relative change between the kinetic derivative and the covariant derivative.

The consequence of the renormalization procedure, besides obtaining explicitly finite results in calculations of physical observables, is that couplings + fields run with energy. This is because the UV structure is the same + ren. conditions are arbitrary for the same theory. Leads to the renormalization group & QFT 2!