Lecture 27.
Quantum Field Theory 1.
Felix Yu
July 14, 2021

Last: Divergences in Electron Vtx Fcn.

Today: Renormalized Perturbation Theory

In this final lecture of QFT1, we will renormalize QED. As emphasized last time, the one-loop diagrams in QED of the electron vertex function, the vacuum polarization of photons, and wavefcn renormalization of fermions are all diagrams with ultraviolet divergences and also infrared divergences. In dimensional regularization, these divergences are expressed mathematically as poles of the Γ-function, which is singular when the argument is 0 or any negative integer. The overriding physical principle when dealing with divergent calculations is physically measurable quantities are always <u>finite</u>, so divergences in calculations must necessarily cancel.

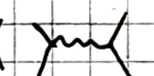
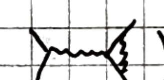
In the case of QFT + ultraviolet divergences, the divergences seen at any given loop order are interpreted as a failure of using the appropriate zeroth-order Lagrangian to perform sensible perturbation theory. This will lead to a reformulation of the "bare" Lagrangian into a "counterterm" Lagrangian and a "renormalized perturbation" Lagrangian. The one-loop divergences in QED will be explicitly matched to counterterms, while the residual + finite one-loop corrections will dictate the explicit & calculable ~~renormalization~~ perturbation Lagrangian correction terms.

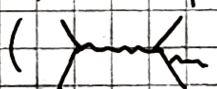
The most intuitive way to think of counterterms are the fact that they "absorb" infinities is to remember that the Lagrangian provides a <u>local</u> description of the degrees of freedom and their interactions. Then, the counterterm provides an "infinite offset" not unlike the shift in the potential height in classical mechanics: just as the <u>difference</u> in initial + final height provided the kinetic energy + there was no physical effect from the absolute magnitude of the initial height of the ball, there is the possibility to perform large (+ possibly infinite) constant shifts in [redacted] Lagrangian parameters without affecting the corresponding dynamics and the dofs. of those dynamics.

Because the cancellation of infinities is mathematically ambiguous, we will adopt <u>renormalization</u> <u>conditions</u> to <u>define</u> the Lagrangian parameters at a given loop order and set the couplings at a certain scale. These renormalization conditions are, in practice, establish as experimental measurements that <u>define</u> <u>the</u> <u>theory</u>. Then, because of renormalization effects (which will be a main focus of QFT2), the Lagrangians of the same theory must agree even when two experiments use [redacted] two different ren. conds. set at two diff. energy scales. [redacted] Intuitively, one set of ren. conds. must <u>flow</u> toward the other as the energy scales get closer together, + they must coincide when the energies match.

Short note on IR divergences:

In general, counterterms & renormalization only deal with UV divergences. But as we ~~saw~~ saw in Compton scattering as well as the electron vtx. fcn., IR divergences can also arise in specific processes. In QED, this typically happens when we have final state photons, which can be emitted a real radiation from charged electrons/positrons with finite momentum. Schematically, there is the IR divergence in the one-loop correction to a cross section that cancels the divergence in the real radiation diagram:

$|\mathcal{M}($  $+$  $)|^2$

$+\ |\mathcal{M}($  $+$  $)|^2 =$ finite,

even though the first $|\mathcal{M}|^2$ & the second $|\mathcal{M}|^2$ are each IR-divergent. Note the cancellation follows the required scaling of QED gauge coupling & charge.

★ The general theorem that IR divergences cancel is from KLN, as mentioned before.

★ IR divergences are <u>real</u>, in that a finite resolution for photon energies gives a cross section that increases logarithmically as the resolution parameter becomes infinitesimal. But the overall cross section is always <u>finite</u>.

The bare QED Lagrangian, counterterms, renormalized Lagrangian, & renormalization conditions.

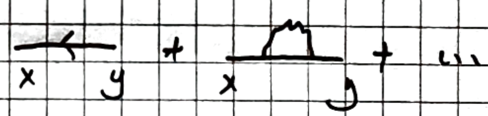
We now relabel our original QED Lagrangian with subscripts to indicate that the fields + couplings are bare parameters:

$$\mathcal{L} = -\frac{1}{4}\left(F_{b\mu\nu}\right)^2 + \bar{\psi}_b\left(i\partial\!\!\!/ - m_0\right)\psi_b - e_0\,\bar{\psi}_b\,\gamma^\mu\psi_b\,A_{\mu b}$$

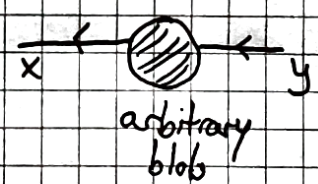
The fields ψ_b + $A_{b\mu}$ are bare fermion & photon fields, which are modified by pert. theory interactions corresponding to wavefcn. renormalization diagrams. To systematize the wavefcn. renormalization, we revisit the 2pt. corr. fcn in interacting theory.

For ψ,

$$\langle\Omega|\,T\psi(x)\bar{\psi}(y)\,|\Omega\rangle =$$

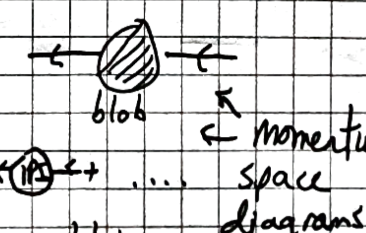


=

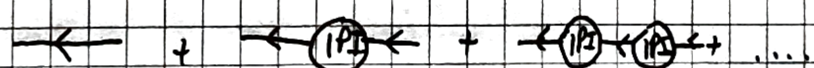


We can factor out an overall $e^{-ip\cdot(x-y)}$ and the common $\int\frac{d^4p}{(2\pi)^4}$ to consider all diagrams instead in momentum space. Then, performing the Fourier transform,

$$\int d^4x\;\langle\Omega|\,T\psi(x)\bar{\psi}(0)\,|\Omega\rangle\, e^{ip\cdot x} =$$



=

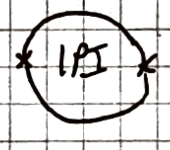


where we have decomposed the arbitrary blob diagramatically into a infinite series of 1PI ~~ins~~ insertions.

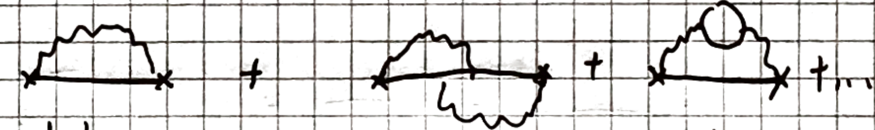
A 1PI diagram is "1-particle irreducible", meaning that the diagram cannot be split into two by removing a single line. Pictorially, we draw an arbitrary diagram contributing to the blob, which is fixed by having two external lines. Then, all loops must terminate on the continuous fermion line. If we can cut the fermion line at any point + separate the diagrams, then we do not have a 1PI diagram.

Now, we perform a power series + loop expansion on 1PI diagrams: We truncate the external propagator lines & label the 1PI diagram as $-i\Sigma(p)$:

$-i\Sigma(p)$ =

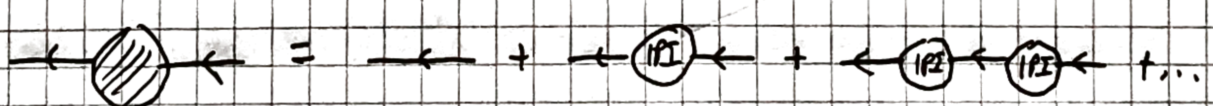


=



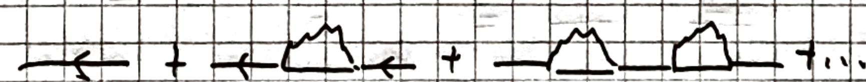
1-loop e^2 "Σ_2" 2-loop, e^4 2-loop, e^4

Then, the blob is approximated by the 1-loop wavefcn. resummation to give a complete 2-pt. correlation fcn. <u>accurate to 1-loop order.</u>



blob

≈



Using our bare pert. theory Lagrangian, we get the RHS as

$$= \frac{i(\not{p}+m_0)}{p^2-m_0^2} + \frac{i(\not{p}+m_0)}{p^2-m_0^2}(-i\Sigma_2)\frac{i\not{p}+m_0}{p^2-m_0^2} + \ldots$$

We can resum this geometric series, since $\Sigma(p)$ ($\approx \Sigma_2(\not{p})$ at one-loop) is a pure number or depends only on $\not{p}$ & so commutes with $\not{p}$:

We can cancel $\dfrac{\not{p}+m_0}{p^2-m_0^2} = \dfrac{\not{p}+m_0}{\not{p}\not{p}-m_0^2} = \dfrac{1}{\not{p}-m_0}$,

so

$$\int d^4x \, \langle \Omega | T(\psi(x)\bar{\psi}(0)) | \Omega \rangle e^{ip\cdot x}$$

$$= \frac{i}{\not{p}-m_0} + \frac{i}{\not{p}-m_0}\left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right) + \frac{i}{\not{p}-m_0}\left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)^2 + \cdots$$

$$= \frac{i}{\not{p}-m_0}\left(1 + \frac{\Sigma(\not{p})}{\not{p}-m_0} + \left(\frac{\Sigma(\not{p})}{\not{p}-m_0}\right)^2 + \cdots\right)$$

$$= \frac{i}{\not{p}-m_0}\left(\frac{1}{1-\frac{\Sigma(\not{p})}{\not{p}-m_0}}\right)$$

$$= \frac{i}{\not{p}-m_0-\Sigma(\not{p})} \approx \frac{i}{\not{p}-m_0-\Sigma_2(\not{p})}$$

, if we evaluate the wavefcn. renorm. to one-loop.

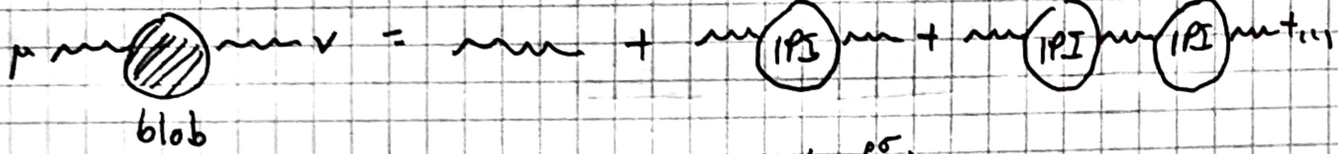
We see that the resummed propagator differs from the bare propagator in two main ways:

1. The mass pole at $p^2 = m_0^2$ is shifted by $\mathrm{Re}[\Sigma_2(\not{p})] \neq 0$.

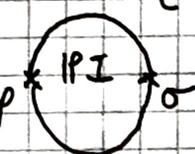
2. The residue of the pole in the denominator is not just "1," but instead is some new factor which we can call Z_2.

In renormalized pert. theory, we will thus require two renormalization conditions to "refix" the fermion propagator at one-loop. We will establish a new fermion field with the propagator pole at a physical mass m and canonical residue of "1," not "Z_2".

In exactly the same way, we will ~~new~~ resum the one-loop diagrams that correct the photon propagator.



$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left(i\Pi^{\rho\sigma}\right)\frac{-ig_{\sigma\nu}}{q^2} + \dots$$

where $i\Pi^{\rho\sigma}(q) \equiv$  with truncated photon lines.

Note that the requirements of Lorentz covariance and the Ward identity fix the form factor structure of the $(\rho\sigma)$ indices: they can only depend on q_ρ, q_σ or $g_{\rho\sigma}$; the vanishing under Ward identity requires

$$i\Pi^{\rho\sigma}(q) = i\left(q^2 g^{\rho\sigma} - q^\rho q^\sigma\right)\Pi(q^2)$$

↳ pure scalar function

The resummation is the same type of geometric series:

$$\text{RHS} = \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left(i\left(q^2 g^{\rho\sigma} - q^\rho q^\sigma\right)\Pi(q^2)\frac{-ig_{\sigma\nu}}{q^2}\right) + \dots$$

$$= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left(\delta^\rho{}_\nu - \frac{q^\rho q_\nu}{q^2}\right)\Pi(q^2)$$

$$+ \frac{-ig_{\mu\rho}}{q^2}\left(\delta^\rho{}_\alpha - \frac{q^\rho q_\alpha}{q^2}\right)\left(\delta^\alpha{}_\nu - \frac{q^\alpha q_\nu}{q^2}\right)\Pi(q^2)^2$$

$$+ \dots$$

Note that $\left(\delta^\rho{}_\alpha - \frac{q^\rho q_\alpha}{q^2}\right)\left(\delta^\alpha{}_\nu - \frac{q^\alpha q_\nu}{q^2}\right) = \delta^\rho{}_\nu - \frac{q^\rho q_\nu}{q^2}$,

so we get

$$\text{RHS} = \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left(\delta^\rho{}_\nu - \frac{q^\rho q_\nu}{q^2}\right)\left(\Pi(q^2) + \Pi^2(q^2) + \dots\right)$$

$$= \frac{-ig_{\mu\nu}}{q^2}\left(\frac{1}{1-\Pi(q^2)}\right) + \frac{iq_\mu q_\nu}{q^4}\left(\Pi(q^2) + \Pi^2(q^2) + \ldots\right)$$

$$= \frac{-ig_{\mu\nu}}{q^2}\left(\frac{1}{1-\Pi(q^2)}\right) - \frac{iq_\mu q_\nu}{q^4} + \frac{iq_\mu q_\nu}{q^4\left(1-\Pi(q^2)\right)}$$

$$= \frac{-i}{q^2}\left(\frac{1}{1-\Pi(q^2)}\right)\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - \frac{i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right)$$

Given that one end of this exact photon propagator is contracted on a fermion line, we can safely drop the terms proportional to q_μ or q_ν by gauge invariance.

So,

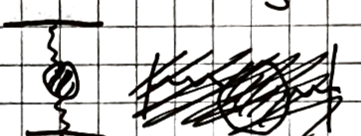
$$= \frac{-ig_{\mu\nu}}{q^2\left(1-\Pi(q^2)\right)}$$

Again, we have two possible effects.

① Mass shift. If ~~$\Pi(q^2)$~~ $\Pi(q^2 \to 0) \neq 0$, then we generate a mass pole for the photon. Fortunately, this never happens.

② The residue change. $Z_3 \equiv \dfrac{1}{1-\Pi(0)}$.

Z_3 is interpreted as charge renormalization, since



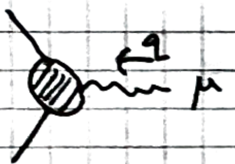
$$= \frac{Z_3 e^2 g_{\mu\nu}}{q^2}$$ is effectively rescaling $e \to e\sqrt{Z_3}$.

The ② residue change will require a ren. condition.

Finally, we have the electron vertex fcn.

At tree-level, the bare coupling just gives a pure γ^μ interaction, but we have seen that the one-loop diagram has both γ^μ and $i\sigma^{\mu\nu}q_\nu$ interaction structures.

In general, we can write (since QED is parity conserving, we do not have $\gamma^\mu\gamma^5$)

blob  $= ie_0 Q\, \Gamma^\mu$,

with $\Gamma^\mu(q) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)$

Again, the form factor structure is determined by Lorentz covariance, and $F_1(q^2)$ & $F_2(q^2)$ are the scalar function form factors. At tree-level, $F_1 = 1$ & $F_2 = 0$.

F_1 = coefficient of γ^μ in n-loop calculation, electric form factor

F_2 = coefficient of $\frac{i\sigma^{\mu\nu}q_\nu}{2m}$ in n-loop calculation, magnetic form factor

We will impose a renormalization condition on $\Gamma^\mu(q)$ to fix the interaction behavior at one choice of q (& make it finite).