Lecture 25.
Quantum Field Theory 1.
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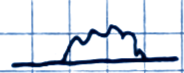
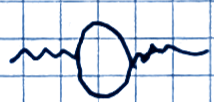
Last time: Finished Compton scattering, One-loop diagram overview in QED.

Today: Electron vertex calculation at one-loop

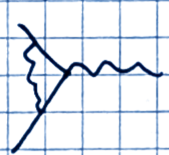
At the end of last time, we discussed one-loop corrections/improvements to external states that appear in legs of hard scattering processes. We had two 2-pt. correlation fns. to calculate at 1-loop: electron wavefcn



and photon wavefunction / vacuum polarization



. We also have the electron vertex function:



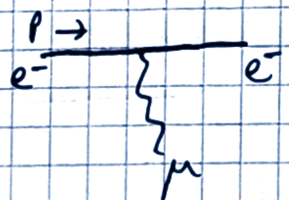
. We start with the electron vertex function.

Since this is our first loop calculation, we will need to introduce new mathematical tools to handle the calculation involving virtual particles. Specifically, the integrals over loop momenta ~~are~~ can be divergent as $|\ell| \to \infty$ and $|\ell| \to 0$. A divergence that occurs when the loop momenta goes to infinity is called an <u>ultraviolet</u> divergence (UV div.), while a divergence that occurs for $|\ell| \to 0$ is called an <u>infrared</u> divergence (IR div.). Loosely speaking, a UV divergence is always above the scale of any experiment at finite energy, & an IR divergence is always below

the scale of any experiment at finite energy.
Thus, we can interpret the UV divergences as the necessity / failure of our model to know about degrees of freedom at smaller length scales / higher energies. We can hope, though, that there is a class of QFTs where the ultraviolet description of the theory is masked / hidden / irrelevant for experiments at lower energy scales. (There is an analogy with atomic physics being insensitive to the quark nature inside nuclei, + many other examples.) In fact, there is such a class of QFTs, which is called <u>renormalizeable</u> theories. Renormalizeable theories are characterized by the fact that all coupling constants either have positive or zero mass dimension (so the operators in the Lagrangian have scaling/canonical dimension $D \leq 4$.)

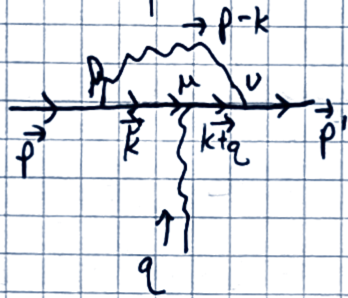
For IR divergences, there is a general theorem, due to Kinoshita, Lee, Nauenberg (KLN) that any physical observable is insensitive to IR divergences. We will explain this concretely when we discuss bremstrahlung + the electron vertex fcn.

In QED, the tree-level vertex for electron interactions with photons is simply



$$= ieQ\gamma^\mu = -ie\gamma^\mu$$

At one-loop, the diagram is



$$k' = k+q$$

$$p+q = p'$$

This diagram gives

$$i\mathcal{M} = \bar{u}(p')\,\Gamma^{\mu}_{1\text{-loop}}(p',p)\,u(p)\,\epsilon_\mu(q)$$

$$= \int \frac{d^4k}{(2\pi)^4}\,\bar{u}(p')\cdot(-ie\gamma^\nu)\,\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}(-ie\gamma^\mu)\,\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}$$

$$(-ie\gamma^\rho)\,u(p)\quad\epsilon_\mu(q)\cdot\frac{-ig_{\nu\rho}}{(p-k)^2+i\epsilon}$$

$$= -e^2\,\epsilon_\mu(q)\int\frac{d^4k}{(2\pi)^4}\,\bar{u}(p')\,\gamma^\nu\,\frac{(\not{k}'+m)\,\gamma^\mu\,(\not{k}+m)\gamma_\nu\,u(p)}{(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)((p-k)^2+i\epsilon)}$$

Use
$$\gamma^\nu\,\not{k}'\,\gamma^\mu\,\not{k}\,\gamma_\nu = -2\,\not{k}\,\gamma^\mu\,\not{k}'$$
$$+\;\gamma^\nu m\gamma^\mu m\gamma_\nu = -2\,m^2\gamma^\mu$$
$$+\;\gamma^\nu\,\not{k}'\,\gamma^\mu m\,\gamma_\nu = 4m\,k'^\mu$$
$$+\;\gamma^\nu\,m\,\gamma^\mu\,\not{k}\,\gamma_\nu = 4m\,k^\mu$$

$$i\mathcal{M} = 2e^2\,\epsilon_\mu(q)\int\frac{d^4k}{(2\pi)^4}\,\frac{\bar{u}(p')\left(\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right)\cdot u(p)}{(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)((p-k)^2+i\epsilon)}$$

We now combine denominators. The goal of combining denominators is to make the overall loop momentum dependence as uniform as possible so we can evaluate by "Gaussian"-style integrals.

As seen in HW 11, the tool we use is the introduction of Feynman parameters:

$$\frac{1}{AB} = \int dx\, dy\ \delta(x+y-1) \frac{1}{(xA+yB)^2}$$

We have three denominators: the identity we need is eqn. 6.41 of P+S.

$$\frac{1}{ABC} = \int_0^1 dx\, dy\, dz\ \delta(x+y+z-1) \frac{2!}{(xA+yB+zC)^3}$$

Note that when we integrate over z (or equivalently, x or y), the remaining limits of integration change:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz\ \delta(x+y+z-1) \ldots$$

$$= \int_0^1 dx \int_0^{1-x} dy \ \ldots \Big|_{z=1-x-y}$$

In our vertex,

$$i\mathcal{M} = 2e^2 \epsilon_\mu(q) \cdot \int_0^1 dx\, dy\, dz \int \frac{d^4k}{(2\pi)^4} \frac{\overset{\delta(x+y+z-1)}{2}\, \bar{u}(p') \left(\not{k}\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu \right) \cdot u(p)}{(xD_1 + yD_2 + zD_3)^3}$$

for
$$D_1 = k'^2 - m^2 + i\epsilon$$
$$D_2 = k^2 - m^2 + i\epsilon$$
$$D_3 = (p-k)^2 + i\epsilon$$

Calculate:
$$\begin{aligned} xD_1 + yD_2 + zD_3 &= x(k+q)^2 - xm^2 + xi\epsilon \\ &\quad + y k^2 - ym^2 + yi\epsilon \\ &\quad + z(p-k)^2 + zi\epsilon \\ &= xk^2 + 2xk\cdot q + xq^2 - xm^2 + i\epsilon \\ &\quad + yk^2 - ym^2 + zp^2 - 2zp\cdot k + zk^2 \\ &= k^2 + 2xk\cdot q - 2zp\cdot k + xq^2 - xm^2 - ym^2 + zp^2 + i\epsilon \end{aligned}$$

We can now shift k to complete the square:

$$= k^2 + 2k\cdot(xq - zp) + xq^2 - xm^2 - ym^2 + zp^2 + i\epsilon$$

$$= (k + (xq - zp))^2 - (xq - zp)^2 + xq^2 - xm^2 - ym^2 + zp^2 + i\epsilon$$

$$\equiv \ell^2 - \Delta + i\epsilon$$

where $\ell \equiv k + xq - zp$

and $\Delta \equiv (xq - zp)^2 - xq^2 + xm^2 + ym^2 - zp^2$

We can simplify these using on-shell $p^2 = m^2$ for external momenta. (Cannot set $q^2 = 0$ since we imagine photon mediates interaction)

$$= x^2 q^2 - 2xz\, q\cdot p + z^2 p^2 - xq^2 + xm^2 + ym^2 - zm^2$$

$$= (x^2 + xz - x)\, q^2 + (z^2 + x + y - z)\, m^2$$

Use $p + q = p'$

$\Rightarrow p^2 + 2p\cdot q + q^2 = (p')^2$

$\Rightarrow 2p\cdot q = -q^2$

$$= x(x + z - 1)\, q^2 + (z^2 + 1 - 2z)\, m^2$$

$$= -xy\, q^2 + (z-1)^2 m^2$$

We have

$$i\mathcal{M} = 2e^2 \epsilon_\mu(q) \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)$$

$$\int \frac{d^4k}{(2\pi)^4} \cdot 2\,\bar{u}(p')\, \frac{\left(\not{k}\,\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right) u(p)}{(\ell^2 - \Delta + i\epsilon)^3}$$

Shift numerator: $k = \ell - xq + zp \qquad k' = k + q = \ell + (1-x)q + zp$

$$= 2e^2 \epsilon_\mu(q) \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)$$

$$\int \frac{d^4\ell}{(2\pi)^4}\, 2\,\bar{u}(p')\, \frac{\left(\not{\ell} - x\not{q} + z\not{p}\right)\gamma^\mu\left(\not{\ell} + (1-x)\not{q} + z\not{p}\right) + m^2\gamma^\mu - 2m(2\ell - 2xq + 2zp + q)^\mu\,)\, u(p)}{(\ell^2 - \Delta + i\epsilon)^3}$$

Multiply out

$$= 2e^2 \epsilon_\mu(q) \int_0^1 dx\,dy\,dz\ \delta(x+y+z-1)$$

$$\int \frac{d^4\ell}{(2\pi)^4} \cdot \frac{2\bar{u}(p') \left(\not{\ell} \gamma^\mu \not{\ell} + \not{\ell}\gamma^\mu((1-x)\not{q} + z\not{p}) + (-x\not{q} + z\not{p})\gamma^\mu \not{\ell} + (-x\not{q}+z\not{p})\gamma^\mu((1-x)\not{q}+z\not{p}) + m^2\gamma^\mu - 2m\,(2\ell - 2xq + 2zp + q)^\mu \right) u(p)}{(\ell^2 - \Delta + i\epsilon)^3}$$

Now, we have the rules for integration:

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\text{odd powers of } \ell}{(\ell^2 - \Delta + i\epsilon)^n} = 0 \quad \text{by oddness}$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\alpha \ell^\beta}{(\ell^2 - \Delta + i\epsilon)^n} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\frac{1}{4} g^{\alpha\beta} \ell^2}{(\ell^2 - \Delta + i\epsilon)^n}$$

But more generally, $\displaystyle\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\alpha \ell^\beta}{(\ell^2 - \Delta + i\epsilon)^n} = \int \frac{d^d\ell}{(2\pi)^d} \frac{\frac{1}{d} g^{\alpha\beta} \ell^2}{(\ell^2 - \Delta + i\epsilon)^n}$

[Pf: vanishes by oddness unless $\alpha = \beta$, check by contracting with $g_{\alpha\beta}$ + requiring covariance.]

Then, $\not{\ell}\gamma^\mu\not{\ell} = \ell^\alpha \ell^\beta \gamma_\alpha \gamma^\mu \gamma_\beta \rightarrow \frac{1}{4} g^{\alpha\beta} \ell^2 \gamma_\alpha \gamma^\mu \gamma_\beta$
$= -\frac{1}{2} \ell^2 \gamma^\mu$

$$= 4e^2 \epsilon_\mu(q) \int_0^1 dx\,dy\,dz\ \delta(x+y+z-1)$$

$$\int \frac{d^4\ell}{(2\pi)^4} \cdot \frac{\bar{u}(p') \left(-\frac{1}{2}\ell^2\gamma^\mu + (-x\not{q}+z\not{p})\gamma^\mu((1-x)\not{q}+z\not{p}) + m^2\gamma^\mu - 2m\,((1-2x)q + 2zp)^\mu \right) u(p)}{(\ell^2 - \Delta + i\epsilon)^3}$$

We can manipulate the numerator further using Dirac equation and anti-commutation of γ-matrices (like Gordon identity).

Note $\not p \gamma^\mu = 2p^\mu - \gamma^\mu \not p$, $\quad \not p\, u(p) = m\, u(p)$

$\not q \gamma^\mu = 2q^\mu - \gamma^\mu \not q$, $\quad \not q\, u(p) = (\not p' - \not p)\, u(p)$

$= (\not p' - m)\, u(p)$

And $\quad \bar u(p')\, \not q\, u(p) = 0$.

Focus on numerator. Set aside $-\frac{1}{2}\ell^2 \gamma^\mu$ term.

$$\bar u(p') \Big(-x(1-x)\not q \gamma^\mu \not q + z(1-x)\not p \gamma^\mu \not q - xz \not q \gamma^\mu \not p + z^2 \not p \gamma^\mu \not p$$
$$+ m^2 \gamma^\mu - 2m(1-2x) q^\mu - 4zm p^\mu \Big) u(p)$$

$$= \bar u(p') \Big(-2x(1-x) q^\mu \cancelto{0}{\not q} + x(1-x)\gamma^\mu q^2$$
$$+ 2z(1-x) p^\mu \cancelto{0}{\not q} - z(1-x)\gamma^\mu \not p \not q$$

$\not p \not q = 2p\cdot q - \not q \not p$

$$- 2xz\, q^\mu \not p + xz\, \gamma^\mu \not q \not p$$
$$+ 2z^2 p^\mu \not p - z^2 \gamma^\mu m^2$$
$$+ m^2 \gamma^\mu - 2m(1-2x) q^\mu - 4zm p^\mu \Big) u(p)$$

$$= \bar u(p') \Big(x(1-x)\gamma^\mu q^2 - 2z(1-x)\gamma^\mu p\cdot q + z(1-x)\gamma^\mu \not q m$$
$$+ xz\gamma^\mu \not q m - 2xz q^\mu m + 2z^2 p^\mu m - z^2 \gamma^\mu m^2$$
$$+ m^2 \gamma^\mu - 2m(1-2x) q^\mu - 4zm p^\mu \Big) u(p)$$

Again, $2p\cdot q = -q^2$

$$= \bar u(p') \Big((x+z)(1-x)\gamma^\mu q^2 + (1-z^2) m^2 \gamma^\mu + z\gamma^\mu \not q m$$
$$+ m\big(q^\mu(-2xz - 2 + 4x)\big) + m(p^\mu)(2z^2 - 4z) \Big) u(p)$$

$$= \bar u(p') \Big((1-y)(1-x)\gamma^\mu q^2 + (1-z^2) m^2 \gamma^\mu + z\gamma^\mu m(\not p' - \not p)$$
$$+ 2m q^\mu(-xz - 1 + 2x) + 2m p^\mu (z)(z-2) \Big) u(p)$$

$$= \bar u(p') \Big((1-y)(1-x)\gamma^\mu q^2 + (1-z-z^2) m^2 \gamma^\mu + zm\gamma^\mu \not p'$$
$$+ 2m q^\mu(-xz - 1 + 2x) + 2mz p^\mu (z-2) \Big) u(p)$$

$$= \bar u(p') \Big((1-y)(1-x)\gamma^\mu q^2 + (1-z-z^2) m^2 \gamma^\mu + 2p'^\mu zm - zm^2\gamma^\mu$$
$$+ 2m q^\mu(-xz - 1 + 2x) + 2mz p^\mu (z-2) \Big) u(p)$$

$$= \bar{u}(p')\Big[(1-y)(1-x)\gamma^\mu q^2 + (1-2z-z^2)m^2\gamma^\mu$$
$$+ 2zm(p+p')^\mu$$
$$+ 2mq^\mu(-xz-1+2x) + 2mzp^\mu(z-3)\Big]u(p)$$

Use $q^\mu = p'^\mu - p^\mu$, can always express momenta as $(p'^\mu + p^\mu)$ and $(p'^\mu - p^\mu) = q^\mu$

$$\Rightarrow \quad 2p^\mu = p'^\mu + p^\mu - (p'^\mu - p^\mu) = p'^\mu + p^\mu - q^\mu$$
$$p^\mu = \tfrac{1}{2}(p'^\mu + p^\mu) - \tfrac{1}{2}q^\mu$$

$$= \bar{u}(p')\Big[(1-y)(1-x)\gamma^\mu q^2 + (1-2z-z^2)m^2\gamma^\mu$$
$$+ \big(2zm + mz(z-3)\big)(p+p')^\mu$$
$$+ mq^\mu(-2xz-2+4x-z^2+3z)\Big]u(p)$$

We get $zm(2+z-3) = zm(z-1)$

$$(-2xz-2+4x-z^2+3z) = (-2xz+2x+z-2y-z^2)$$
$$= \big(z(1-2x-z)+2(x-y)\big)$$
$$= z(-x+y)+2(x-y)$$
$$= (2-z)(x-y)$$

Since rest of integral is symmetric for $x \leftrightarrow y$ but coeff. of q^μ is not, we get coeff. of $q^\mu \to 0$ after $\int dx\,dy\,dz\,\delta(x+y+z-1)$ integration, as required by Ward identity

Restoring ℓ^2 term:

$$i\mathcal{M} = 4e^2\,\epsilon_\mu(q)\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)$$
$$\int \frac{d^4\ell}{(2\pi)^4}\,\frac{\bar{u}(p')\Big(-\tfrac{1}{2}\ell^2\gamma^\mu + (1-y)(1-x)\gamma^\mu q^2 + (1-2z-z^2)m^2\gamma^\mu + zm(z-1)(p+p')^\mu\Big)u(p)}{(\ell^2-\Delta+i\epsilon)}$$

Next, use Gordon identity to eliminate $(p+p')^\mu$ for $\sigma^{\mu\nu} q_\nu$.

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p)$$

$$i\mathcal{M} = 4e^2 \, \epsilon_\mu(q) \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1)$$

$$\int \frac{d^4\ell}{(2\pi)^4} \cdot \frac{\bar{u}(p') \Big(-\frac{1}{2} \ell^2 \gamma^\mu + (1-y)(1-x) \gamma^\mu q^2 + (1-4z+z^2) m^2 \gamma^\mu + z m (z-1)\, i\sigma^{\mu\nu} q_\nu \Big) u(p)}{(\ell^2 - \Delta + i\epsilon)}$$

These Lorentz structures are guaranteed just by the Ward identity and Lorentz covariance & momentum conservation.

Next time: Dimensional regularization, Wick rotation, QED vertex form factors.