Lecture 24.

Quantum Field Theory I.

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Last time: Compton scattering.

Today: Finish Compton scattering. One-loop diagrams in QED

Correction: $u = (p_1 - p_4)^2 = 2m_e^2 - 2p_1\cdot p_4 = (p_3 - p_2)^2 = -2p_2\cdot p_3$

From last time:

$$\frac{1}{4}\sum_{\substack{\text{spins,}\\ \text{pols.}}} |\mathcal{M}|^2 = e^4\Bigg[\frac{1}{(p_1\cdot p_2)^2}\Big(2p_1\cdot p_2\, p_2\cdot p_4 + 2m_e^2(2p_1\cdot p_2 - p_2\cdot p_4 - p_1\cdot p_4) + 4m_e^4\Big)$$
$$- \frac{1}{p_1\cdot p_2\, p_1\cdot p_3}\Big(2m_e^2(-p_1\cdot p_3 + p_1\cdot p_2) + 4m_e^4\Big)$$
$$+ \frac{1}{(p_1\cdot p_3)^2}\Big(2p_1\cdot p_3\, p_3\cdot p_4 + 2m_e^2(-2p_1\cdot p_3 + p_3\cdot p_4 - p_1\cdot p_4) + 4m_e^4\Big)\Bigg]$$

We first simplify further

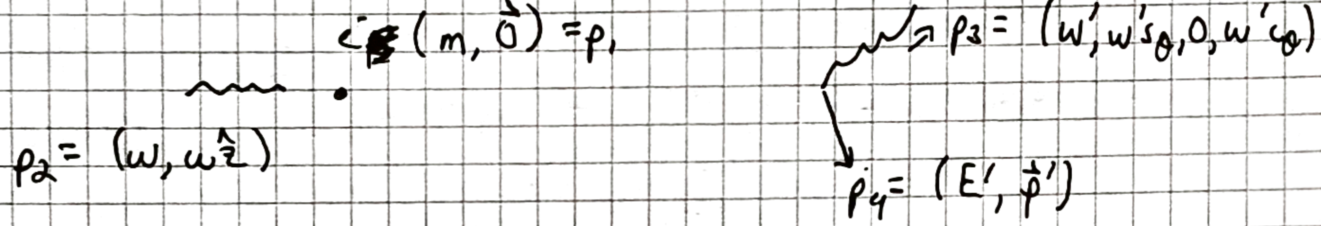
$$\frac{1}{4}\sum|\mathcal{M}|^2 = e^4\Bigg[\frac{1}{p_1 p_2}\big(2p_2\cdot p_4\big) + \frac{2m_e^2}{(p_1\cdot p_2)^2}\big(2p_1\cdot p_2 - p_1\cdot p_3 - p_1\cdot p_4\big) + \frac{4m_e^4}{(p_1\cdot p_2)^2}$$
$$+ \frac{2m_e^2}{p_1\cdot p_2} - \frac{2m_e^2}{p_1\cdot p_3} - \frac{4m_e^4}{p_1\cdot p_2\, p_1\cdot p_3}$$
$$+ \frac{2p_3\cdot p_4}{p_1\cdot p_3} + \frac{2m_e^2}{(p_1\cdot p_3)^2}\big(-2p_1\cdot p_3 + p_1\cdot p_2 - p_1\cdot p_4\big) + \frac{4m_e^4}{(p_1\cdot p_3)^2}\Bigg]$$

Use $p_2\cdot p_4 = p_1\cdot p_3$, $p_3\cdot p_4 = p_1\cdot p_2$,

$$= e^4\Bigg[\frac{2p_1\cdot p_3}{p_1\cdot p_2} + \frac{2p_1\cdot p_2}{p_1\cdot p_3} - \cancel{\frac{2m_e^4}{(p_1\cdot p_2)^2}} + \frac{\overset{4}{\cancel{2}}m_e^2}{p_1\cdot p_2} + \frac{\overset{2}{\cancel{4}}m_e^4}{(p_1\cdot p_2)^2}$$
$$+ \cancel{\frac{2m_e^2}{p_1\cdot p_2}} - \cancel{\frac{2m_e^2}{p_1\cdot p_3}} - \frac{4m_e^4}{p_1\cdot p_2\, p_1\cdot p_3} - \cancel{\frac{2m_e^4}{(p_1\cdot p_3)^2}} - \frac{\overset{4}{\cancel{2}}m_e^2}{p_1\cdot p_3} + \frac{\overset{2}{\cancel{4}}m_e^4}{(p_1\cdot p_3)^2}\Bigg]$$

$$\sum \frac{1}{4}|\mathcal{M}|^2 = 2e^4\left(\frac{p_1\cdot p_3}{p_1\cdot p_2} + \frac{p_1\cdot p_2}{p_1\cdot p_3} + 2m_e^2\left(\frac{1}{p_1\cdot p_2} - \frac{1}{p_1\cdot p_3}\right) + m_e^4\left(\frac{1}{p_1\cdot p_2} - \frac{1}{p_1\cdot p_3}\right)^2\right)$$

Again, for the cross section, we specify a frame for the kinematics. Easiest to consider an initial electron at rest & incoming photon with momentum in $\hat{z}$ direction.



Use standard variables ω & θ as the relevant degrees of freedom.

$$m^2 = p_4^2 = (p_1+p_2-p_3)^2 = p_1^2 + 2p_1\cdot p_2 + 0 - 2p_2\cdot p_3 + 0 - 2p_1\cdot p_3$$

$$m^2 = m^2 + 2m\omega - 2m\omega' - 2\omega\omega' + 2\omega\omega' c_\theta$$

$$\Rightarrow \quad 2m(\omega' - \omega) = -2\omega\omega'(1 - c_\theta)$$

$$\Rightarrow \quad -\frac{1}{\omega} + \frac{1}{\omega'} = \frac{1}{m}(1 - c_\theta)$$

This is Compton's formula for the shift of the photon energy.

We also solve for
$$\omega' = \left(\frac{1}{\omega} + \frac{1}{m}(1-c_\theta)\right)^{-1}$$
$$= \left(\frac{m + \omega(1-c_\theta)}{m\omega}\right)^{-1}$$
$$\omega' = \frac{m\omega}{m + \omega(1-c_\theta)} = \frac{\omega}{1 + \frac{\omega}{m}(1-c_\theta)}$$

We also calculate the phase space:

$$\int d\Pi_2 = \int \frac{d^3p_3}{(2\pi)^3}\frac{1}{2\omega'}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E'}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)$$

$$= \int \frac{\omega'^2 d\omega' d\Omega}{(2\pi)^3}\cdot\frac{1}{4\omega' E'}\cdot 2\pi\,\delta(E' + \omega' - \omega - m)$$

with $E' = \sqrt{|\vec{p}'|^2 + m^2} = \sqrt{m^2 + \omega^2 + \omega'^2 - 2\omega\omega' c_\theta}$

Use $\vec{p}' + \vec{p}_3 = \vec{p}_2$

$\vec{p}' = \vec{p}_2 - \vec{p}_3$

$|\vec{p}'|^2 = |\vec{p}_2|^2 + |\vec{p}_3|^2 - 2|\vec{p}_2||\vec{p}_3|\cos\theta$

As usual, $\displaystyle\int dx\, \delta(f(x))\, g(x) = \sum_{x_0} \frac{g(x)}{|f'(x)|}\Bigg|_{x=x_0}$

So, $f(\omega') = E' + \omega' - \omega - m,$

$$\frac{d}{d\omega'} f(\omega') = 1 + \frac{dE'}{d\omega'}$$

$$= 1 + \frac{1}{2E'}\cdot\left(2\omega' - 2\omega c_\theta\right) = 1 + \frac{\omega' - \omega c_\theta}{E'}$$

So, $\displaystyle\int d\Pi_2 = \int dc_\theta \cdot \frac{2\pi}{(4\pi^2)}\, \frac{\omega'}{4E'} \cdot \frac{1}{\left|1 + \frac{\omega' - \omega c_\theta}{E'}\right|}\Bigg|_{\omega' \text{ from Compton's formula}}$

$$= \int \frac{dc_\theta}{(2\pi)} \cdot \frac{\omega'}{4} \frac{1}{|E' + \omega' - \omega c_\theta|}$$

But by energy conservation, $E' + \omega' = \omega + m$

$$= \int \frac{dc_\theta}{8\pi} \frac{\omega'}{(m + \omega(1 - c_\theta))}, \quad \text{use } \frac{\omega'}{m\omega} = \frac{1}{m + \omega(1 - c_\theta)}$$

$$= \int \frac{dc_\theta}{8\pi} \cdot \frac{\omega'^2}{m\omega}$$

Apply diff. cross section formula. $\dfrac{1}{|v_A - v_B|} = 1$

$$\frac{d\sigma}{dc_\theta} = \frac{1}{2\omega}\frac{1}{2m}\cdot\frac{1}{8\pi}\cdot\frac{(\omega')^2}{\omega m}\cdot\left(\frac{1}{4}\sum_{\text{spins}} |M|^2\right)$$

$$= \frac{1}{32\pi}\frac{(\omega')^2}{\omega^2 m^2}\left(2e^4\right)\left(\frac{p_1\cdot p_3}{p_1\cdot p_2} + \frac{p_1\cdot p_2}{p_1\cdot p_3} + 2m_e^2\left(\frac{1}{p_1\cdot p_2} - \frac{1}{p_1\cdot p_3}\right) + m_e^4\left(\frac{1}{p_1\cdot p_2} - \frac{1}{p_1\cdot p_3}\right)^2\right)$$

Use $p_1\cdot p_2 = m\omega$, $p_1\cdot p_3 = \omega' m$

$$\frac{d\sigma}{dc_\theta} = \frac{1}{16\pi} e^4 \cdot \frac{(\omega')^2}{m^2\omega^2}\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2m_e^2\left(\frac{1}{m\omega} - \frac{1}{m\omega'}\right) + m_e^4\left(\frac{1}{m\omega} - \frac{1}{m\omega'}\right)^2 \right)$$

$$= \pi\alpha^2 \left(\frac{\omega'}{m\omega}\right)^2 \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2m\left(\frac{1}{m}\right)(c_\theta - 1) + m^2\left(\frac{1}{m^2}\right)(c_\theta - 1)^2 \right)$$

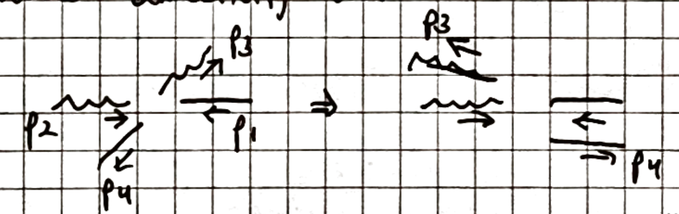
$$= \pi\alpha^2 \left(\frac{\omega'}{m\omega}\right)^2 \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2c_\theta - 2 + c_\theta^2 + 1 - 2c_\theta \right)$$

$$\frac{d\sigma}{dc_\theta} = \pi\alpha^2 \left(\frac{\omega'}{m\omega}\right)^2 \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - s_\theta^2 \right)$$

This is the Klein-Nishina formula.

High-energy behavior:

Change to CoM frame, photon becomes preferred in backwards direction, with



$$\frac{1}{4}\sum |M|^2 \approx 2e^4 \frac{p_1\cdot p_2}{p_1 \cdot p_3} \approx 2e^4 \frac{E+\omega}{E+\omega c_\theta}$$

$$\frac{d\sigma}{dc_\theta} \approx \frac{1}{2}\cdot\frac{1}{2E}\frac{1}{2\omega}\cdot\frac{\omega}{2\pi\, 4(E+\omega)} \frac{2e^4(E+\omega)}{E+\omega c_\theta}$$

$$\approx \frac{2\pi\alpha^2}{2m^2 + s(1+c_\theta)}$$

Note, finite electron mass cuts off singularity as $\theta \to \pi$.

Treat integral as cutoff:

$$\int_{-1}^{1} dc_\theta \frac{d\sigma}{dc_\theta} \approx \frac{2\pi\alpha^2}{s}\int_{-1+2m^2/s}^{1} dc_\theta \cdot \frac{1}{1+c_\theta}$$

$$\Rightarrow \sigma_{tot} = \frac{2\pi\alpha^2}{s}\log\left(\frac{s}{m^2}\right).$$

Can also analyze as consequence of angular momentum conservation.

We will see that singularities of this type are prevalent throughout QED.

Will skip pair annihilation calculation explicitly.
Only mention that it is related by crossing symmetry to Compton scattering. See p. 168-169 of P+S.

We have now dealt with a fair number of tree-level diagrams in QED: technicalities include

1. external (anti-) fermions, fermion flow
2. fermion propagators
3. photon propagators
4. external photons.
5. relative signs between related diagrams.
6. Mandelstam variables
7. Crossing symmetry
8. Ward identity.
   (Can always replace ext. photon pol. vector by its 4-momentum and check gauge invariance.)

Now, we have to consider the one-loop diagrams in QED.

Recall the QED Lagrangian and our interpretation of matrix elements in terms of Feynman diagrams:

$$\mathcal{L} = i\bar{\Psi}\not{D}\Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad D_\mu \equiv \partial_\mu - ieQA_\mu$$

For given process, $i\mathcal{M} = \sum$ all connected, amputated diagrams with given ext. particles

When we speak of one-loop corrections, we generally distinguish between ~~of~~ processes that are accurate to one-loop order and external states that are accurate to one-loop order. In fact, we will find that the only self-consitent approach is when we reinterpret all distinct terms in the Lagrangian as being corrections of external states and their ~~pr~~ interaction properties at one-loop order, after which we can then calculate any given QED process (with amputated legs for external states) at one-loop order self-consistently.

The QED Lagrangian has 4 distinct properties to be calculated at one-loop order:

$\bar{\psi} i \not{\partial} \psi$ → "wavefunction renormalization" of fermion ψ

i.e. Change to Green's fcn / overall propagator

$-m\bar{\psi}\psi$ ⇒ "mass renormalization" of fermion ψ

i.e. Shift in pole mass / pole of Green's fcn.

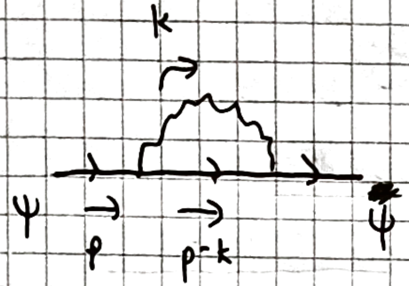
$-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ → "wavefcn renormalization" of photon A_μ

i.e. Change to Green's fcn / overall propagator

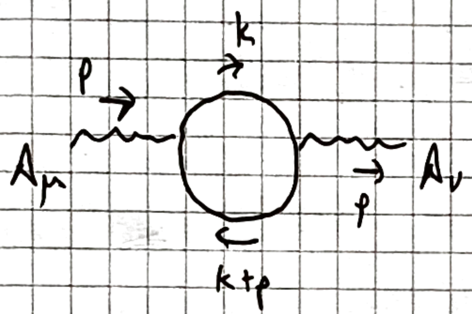
$\bar{\psi} e Q \not{A} \psi$ → "charge/vertex renormalization"

i.e. Change to interaction structure & coupling strength

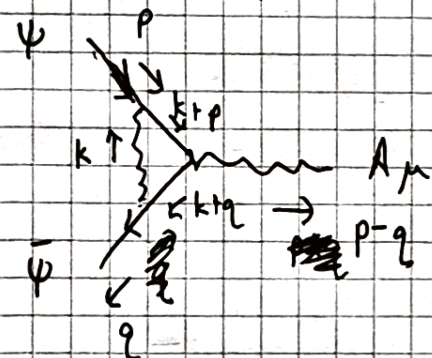
In terms of Feynman diagrams, there are 3 Feynman diagrams that determine the above corrections.



2-pt. corr. fcn of ψ @ 1 loop,
gives WFR + mass ren. of ψ



2-pt. corr. fcn. of A_μ @ 1-loop,
gives WFR of A_μ

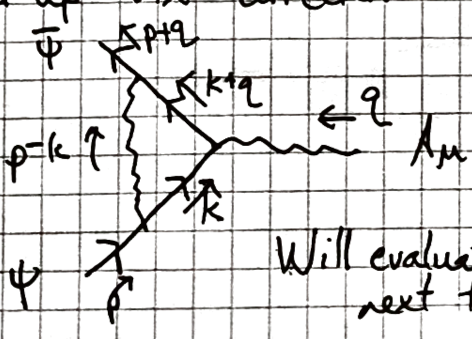


vertex fcn. gives charge ren. @ 1-loop.

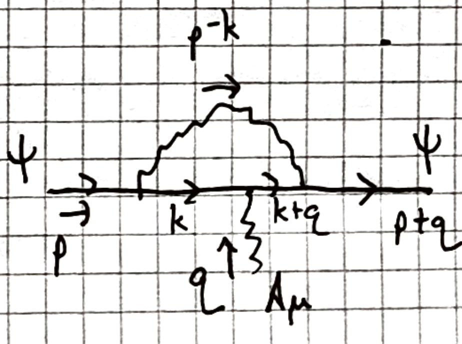
Formally, the separation between amputated diagrams + external leg corrections is ensured by the LSZ reduction formula. Intuitively, all we need is a procedure to match an asymptotic state with momentum p + given spin/pol. + internal quantum numbers to an external leg with momentum p + given spin/pol. + same quantum numbers.

The LSZ reduction formula guarantees that these states are given as poles, hence the necessity of calculating the above 2 pt. corr. fcns.

Set up vtx correction



or:



Will evaluate next time.