Lecture 21.
Quantum Field Theory I.
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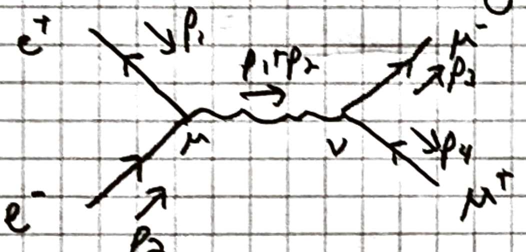
Last time: QED Feynman rules & $U(1)$ gauge symmetry.
Today: $e^+e^- \to \mu^+\mu^-$ scattering.

We continue with the tree-level QED process of $e^+e^- \to \mu^+\mu^-$ scattering. There is only one diagram,



$$i\mathcal{M} = \bar{u}^s(p_3)(-ie\gamma^\nu)v^{s'}(p_4)\left(\frac{-ig_{\mu\nu}}{(p_1+p_2)^2+i\epsilon}\right)$$

$$\bar{v}^r(p_1)(-ie\gamma^\mu)u^{r'}(p_2)$$

$$= \frac{ie^2}{(p_1+p_2)^2+i\epsilon}\bar{u}^s(p_3)\gamma_\mu v^{s'}(p_4)\,\bar{v}^r(p_1)\gamma^\mu u^{r'}(p_2)$$

Drop $i\epsilon$ since tree-level.
Later, spin indices on spinors will be implicit & also dropped.

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1+p_2)^4}\left(\bar{u}^s(p_3)\gamma_\mu v^{s'}(p_4)\right)\left(\bar{v}^r(p_1)\gamma^\mu u^{r'}(p_2)\right)$$
$$\left(\bar{v}^{s'}(p_4)\gamma_\nu u^s(p_3)\right)\left(\bar{u}^{r'}(p_2)\gamma^\nu v^r(p_1)\right)$$

Note: Have to introduce new dummy index ν for second $\mathcal{M}^\dagger$, since each $\mathcal{M}$ is its own separate sum on Lorentz indices.

Note 2: We can freely commute these indexed spin contractions $\left(\bar{u}^s(p_3)\,\gamma_\mu\, v^{s'}(p_4)\right)$ or $\left(\bar{v}^{s'}(p_4)\,\gamma_\nu\, u^s(p_3)\right) = \left(\bar{v}\,\gamma_\nu\, u\right)^{s's}$ since we can consider it as an indexed number of the corresponding fermion bilinear, indexed by $s + s'$ or $r + r'$. So, we can rearrange $|\mathcal{M}|^2$ to be

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1+p_2)^4} \left(\bar{u}^s(p_3)\gamma_\mu v^{s'}(p_4)\right)\left(\bar{v}^{s'}\gamma_\nu u^s(p_3)\right)$$
$$\left(\bar{v}^r(p_1)\gamma^\mu u^{r'}(p_2)\right)\left(\bar{u}^{r'}(p_2)\gamma^\nu v^r(p_1)\right)$$

This rearrangement is reflective of the fact that the matrix element is built from "independent" continuous fermion lines, & it does not matter what order you write down the fermion spin contractions.

Now, given that we want to calculate an unpolarized cross section, we can sum over all initial state & final state spins and average over each of the two initial state spins:

$$\frac{1}{2}\,\frac{1}{2}\sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{e^4}{4(p_1+p_2)^4} \sum_{s,s',r,r'} (\quad)(\quad)(\quad)(\quad)$$

Note, however, that we can rearrange the fermion bilinears & take advantage of a very useful identity:

$$\sum_{s,s'} \left(\bar{u}^s(p_3)\,\gamma_\mu\, v^{s'}(p_4)\;\; \bar{v}^{s'}\,\gamma_\nu\, u^s(p_3)\right)$$
$$= \sum_{s,s'} \bar{u}^s_a(p_3)(\gamma_\mu)_{ab}\, v^{s'}_b(p_4)\;\; \bar{v}^{s'}_c(\gamma_\nu)_{cd}\, u^s_d(p_3)$$

with a,b,c,d the explicit spinor indices

$$= \sum_{s,s'} (\gamma_\mu)_{ab}\, v^{s'}_b(p_4)\, \bar{v}^{s'}_c(p_4)\, (\gamma_\nu)_{cd}\, u^s_d(p_3)\, \bar{u}^s_a(p_3)$$

Use the identity (recall HW #9),

$$\sum_s u^s(p)\, \bar{u}^s(p) = \not{p} + m \mathbb{1}$$

$$\sum_s v^s(p)\, \bar{v}^s(p) = \not{p} - m \mathbb{1}$$

$$= (\gamma_\mu)_{ab} \cdot (\not{p}_4 - m_\mu)_{bc} \cdot (\gamma_\nu)_{cd}\, (\not{p}_3 + m_\mu)_{da}$$

$$= \mathrm{Tr}\left[\gamma_\mu (\not{p}_4 - m_\mu) \gamma_\nu (\not{p}_3 + m_\mu)\right]$$

Similarly, $\sum_{r,r'} \left(\bar{v}^r(p_1)\gamma^\mu u^{r'}(p_2)\right)\left(\bar{u}^{r'}(p_2)\gamma^\nu v^r(p_1)\right)$

$$= \mathrm{Tr}\left[\gamma^\mu (\not{p}_2 + m_e) \gamma^\nu (\not{p}_1 - m_e)\right]$$

This simple rearrangement to traces of Dirac matrices is generic for unpolarized cross sections, since the sum over fermion spins serves to consistently contract all spinor indices.

Thus, we have

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4(p_1+p_2)^4}\, \mathrm{Tr}\left[\gamma_\mu(\not{p}_4 - m_\mu)\gamma_\nu(\not{p}_3 + m_\mu)\right]$$
$$\mathrm{Tr}\left[\gamma^\mu(\not{p}_2 + m_e)\gamma^\nu(\not{p}_1 - m_e)\right]$$

Now, recall the trace identities:

$\mathrm{Tr}[\gamma^{\mu_1} \ldots \gamma^{\mu_n}] = 0$ for odd n

$\mathrm{Tr}[\gamma^{\mu_1} \ldots \gamma^{\mu_n}\gamma^5] = 0$ for odd n

$\mathrm{Tr}[\gamma^5] = 0$

$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^5] = 0$

$\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2} \ldots \gamma^{\mu_n}] = \mathrm{Tr}[\gamma^{\mu_n} \ldots \gamma^{\mu_2}\gamma^{\mu_1}]$ trace reversal

$\mathrm{Tr}[\gamma^{\mu_1} \ldots \gamma^{\mu_n}] = \mathrm{Tr}[\gamma^{\mu_2} \ldots \gamma^{\mu_n}\gamma^{\mu_1}]$ cyclic

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu}$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

(+ many other identities)

We can evaluate the two traces independently

$$\mathrm{Tr}[\gamma_\mu (\not{p}_4 - m_\mu)\gamma_\nu (\not{p}_3 + m_\mu)]$$

$$= \mathrm{Tr}[\gamma_\mu (\not{p}_4)\gamma_\nu \not{p}_3] - \mathrm{Tr}[m_\mu^2 \gamma_\mu \gamma_\nu]$$

since other terms will have odd # of γ-matrices inside trace

Now, note $\not{p}_4 \equiv p_4^\alpha \gamma_\alpha$, $\not{p}_3 \equiv p_3^\beta \gamma_\beta$, with dummy indices α + β.

So $\mathrm{Tr}[\gamma_\mu \not{p}_4 \gamma_\nu \not{p}_3] = p_4^\alpha p_3^\beta \, \mathrm{Tr}[\gamma_\mu \gamma_\alpha \gamma_\nu \gamma_\beta]$

$$= p_4^\alpha p_3^\beta \cdot 4\,(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha})$$

$$= 4(p_{4\mu}p_{3\nu} - g_{\mu\nu}\, p_3 \cdot p_4 + p_{3\mu}p_{4\nu})$$

As you get more comfortable with these traces, it is simpler to immediately go from beginning to end,

$$\mathrm{Tr}[\gamma_\mu \not{p}_4 \gamma_\nu \not{p}_3] = 4[p_{4\mu}p_{3\nu} - g_{\mu\nu}\, p_3 \cdot p_4 + p_{3\mu}p_{4\nu}),$$

since the contractions of the metric tensor can usually be read off.

We also have $-\mathrm{Tr}[m_\mu^2 \gamma_\mu \gamma_\nu] = -4m_\mu^2 g_{\mu\nu}$

And $\mathrm{Tr}[\gamma^\mu (\not{p}_2 + m_e)\gamma^\nu (\not{p}_1 - m_e)]$

$$= 4(p_2^\mu p_1^\nu - g^{\mu\nu}\, p_1 \cdot p_2 + p_1^\mu p_2^\nu - m_e^2 g^{\mu\nu})$$

So, $\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{e^4}{4(p_1+p_2)^4}\, 16\left(p_{4\mu}p_{3\nu} - g_{\mu\nu}\, p_3\cdot p_4 + p_{3\mu}p_{4\nu} - m_\mu^2 g_{\mu\nu}\right)$

$$\left(p_2^\mu p_1^\nu - g^{\mu\nu}\, p_1\cdot p_2 + p_1^\mu p_2^\nu - m_e^2 g^{\mu\nu}\right)$$

Multiply out:

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{4e^4}{(p_1+p_2)^4}\cdot\Big(p_2\cdot p_4\, p_1\cdot p_3 - p_1\cdot p_2\, p_3\cdot p_4 + p_1\cdot p_4\, p_2\cdot p_3 - m_e^2\, p_3\cdot p_4$$
$$- p_1\cdot p_2\, p_3\cdot p_4 + p_1\cdot p_2\, p_3\cdot p_4\, g_{\mu\nu}g^{\mu\nu} - p_1\cdot p_2\, p_3\cdot p_4$$
$$+ m_e^2\, p_3\cdot p_4\, g_{\mu\nu}g^{\mu\nu}$$
$$+ p_2\cdot p_3\, p_1\cdot p_4 - p_1\cdot p_2\, p_3\cdot p_4 + p_1\cdot p_3\, p_2\cdot p_4$$
$$- m_e^2\, p_3\cdot p_4$$
$$- m_\mu^2\left(p_1\cdot p_2 - p_1\cdot p_2\, g_{\mu\nu}g^{\mu\nu} + p_1\cdot p_2 - m_e^2 g_{\mu\nu}g^{\mu\nu}\right)\Big)$$

Note: $g_{\mu\nu}g^{\mu\nu} = g_\mu{}^\mu = \mathrm{Tr}[g_\mu{}^\nu] = 4$ (not $\mathrm{Tr}[g_{\mu\nu}] = -2$)

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{4e^4}{(p_1+p_2)^4}\Big(2\, p_1\cdot p_3\, p_2\cdot p_4 + 2\, p_1\cdot p_4\, p_2\cdot p_3 + 2m_e^2\, p_3\cdot p_4 + 2m_\mu^2\, p_1\cdot p_2 + 4m_\mu^2 m_e^2\Big)$$

We typically evaluate cross sections in the center of mass frame, where the net 3-momentum of all particles is $\vec{0}$ before & after the collision.

Also, without loss of generality, we can orient the two incoming electrons along the $\hat{z}$ axis.

For simplicity, we can set $m_e = 0$, then the electron 4-momenta are lightlike.

Convention: $\left[p = (E, p_x, p_y, p_z)\right]$

$p_1 = (E, 0, 0, E)$, $p_2 = (E, 0, 0, -E)$

$p_3 = (E_\mu, \vec{k})$, $p_4 = (E_\mu, -\vec{k})$

Recall that 2-to-2 kinematics is entirely fixed (allowing for the COM frame + azimuthal rotation).

Cons. of E: $2E = 2E_\mu \Rightarrow E = E_\mu$

Also, $E_\mu = \sqrt{|\vec{k}|^2 + m_\mu^2} = E \Rightarrow |\vec{k}|^2 = E^2 - m_\mu^2$

Evaluate the dot products:

$$p_1 \cdot p_3 = E^2 - E k_z = E^2 - E|\vec{k}|\cos\theta$$
$$p_2 \cdot p_4 = E^2 - E|\vec{k}|\cos\theta$$
$$p_1 \cdot p_4 = E^2 + E|\vec{k}|\cos\theta$$
$$p_2 \cdot p_3 = E^2 + E|\vec{k}|\cos\theta$$
$$p_1 \cdot p_2 = 2E^2$$
$$(p_1+p_2)^2 = 4E^2.$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{16E^4}\Big(E^2(E - |\vec{k}|\cos\theta)^2 + E^2(E+|\vec{k}|\cos\theta)^2 + 2E^2 m_\mu^2 \Big)$$

$$= \frac{e^4}{2E^2}\left(2E^2 + 2|\vec{k}|^2\cos^2\theta + 2m_\mu^2\right)$$

$$= \frac{e^4}{E^2}\left(E^2 + m_\mu^2 + (E^2 - m_\mu^2)\cos^2\theta\right)$$

$$= e^4\left(\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right)$$

Finally, use

from avg. of initial spins

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\vec{p}_3|}{(2\pi)^2 4E_{CM}} \left(\frac{1}{4}\sum\right)|\mathcal{M}|^2$$

where $|v_A - v_B| = \left|\frac{p_{zA}}{E_A} - \frac{p_{zB}}{E_B}\right| = 2$, $E_{CM} = 2E$,

$$\sigma = \int d\Omega \cdot \frac{1}{4 \cdot E^2 \cdot 2} \cdot \frac{\sqrt{E^2 - m_\mu^2}}{(4\pi^2) \cdot 4 \cdot 2E} \cdot e^4 \left(\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right)$$

$$= \frac{2\pi}{256\,\pi^2 E^2} \int d\cos\theta \, \sqrt{1-(m_\mu^2/E^2)}\; e^4 \left(\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right)$$

Use $\alpha = \frac{e^2}{4\pi}$

$$= \frac{1}{128\pi E^2} 16\pi^2 \alpha^2 \sqrt{1-(m_\mu^2/E^2)} \int_{-1}^{1} d\cos\theta \left(\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right)$$

$$= \frac{\pi\alpha^2}{8E^2} \left(2 + \frac{2m_\mu^2}{E^2} + \frac{2}{3} - \frac{2}{3}\frac{m_\mu^2}{E^2} \right) \sqrt{1 - \frac{m_\mu^2}{E^2}}$$

$$= \frac{\pi\alpha^2}{8E^2} \left(\frac{8}{3} + \frac{4}{3}\frac{m_\mu^2}{E^2} \right) \sqrt{1 - \frac{m_\mu^2}{E^2}}$$

$$= \frac{\pi\alpha^2}{3E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{1}{2}\frac{m_\mu^2}{E^2} \right)$$

If we use $E = \frac{E_{CM}}{2}$, we get

$$\sigma_{tot} = \frac{4\pi\alpha^2}{3E_{CM}^2} \sqrt{1 - \frac{4m_\mu^2}{E_{CM}^2}} \left(1 + \frac{2m_\mu^2}{E_{CM}^2} \right)$$

Remark: The cross section expression is invalid for $E_{CM} < 2m_\mu$, which is the threshold energy for producing two muons with no momentum in the CM frame.

Dimensionally, $[\sigma]$ = area = $[\text{energy}]^{-2}$, which is correct.

We can also immediately tell that $\sigma \propto e^4 = \alpha^2$, just by counting the coupling constants in $\mathcal{M}$ and knowing the cross section is prop. to $|\mathcal{M}|^2$.

We can reuse the same calculation for a very important process,

$e^+e^- \to q\bar{q}$, where q = quarks, $\bar{q}$ = antiquarks of the SM. Will start with this next time.

Will also do Compton scattering.

Steps: From $\mathcal{M}$ to σ.

1. Draw diagrams @ given coupling order + loop/tree order.
2. Translate to $\mathcal{M}$ expressions.
3. Square + sum over final spins, avg. over initial spins.
4. Evaluate traces.
5. Find convenient refrence frame for evaluating kinematics. (Usually CM frame, sometimes one particle @ rest frame.)
6. Integrate over phase space.