Lecture 20.
Quantum Field Theory 1.
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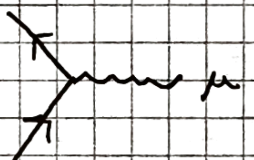
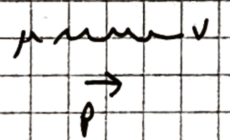
Last time: Feynman rules for fermions, Yukawa theory
Today: QED.

At the end of last time, we introduced the Feynman rule for quantum electrodynamics; the interaction between a fermion with charge Q and the photon is

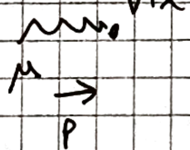


$$= +iQe\gamma^\mu.$$

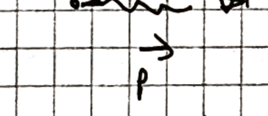
We also need the Feynman rules for vector particles (specifically, Abelian gauge theory):



$$= \frac{-ig_{\mu\nu}}{p^2+i\epsilon} \quad \text{photon propagator}$$



$$= \epsilon_\mu(p) \quad \text{incoming photon}$$



$$= \epsilon^*_\mu(p) \quad \text{outgoing photon}$$

We will not derive ~~those~~ the propagator rule~~s~~ (~~they are~~ it is generally derived after introducing functional methods + path integral formulation of QFT). But note ϵ_μ is a complex 4-vector denoting the polarization of the photon.

We can understand the rules for external vector particles if we write the field equation for A_μ in Lorenz gauge ($\partial_\mu A^\mu = 0$), which ~~p~~ preserves relativistic invariance.

$$\partial^2 A_\mu = 0.$$

Then, each component of A_μ can be expanded as a plane wave, while the overall solution must be a four-vector. We get

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{\sqrt{2E_p}}\right) \sum_{r=0}^{3} \left(a_p^r \epsilon_\mu^r(p) e^{-ip\cdot x} + a_p^{r\dagger} \epsilon_\mu^{r*}(p) e^{+ip\cdot x} \right)$$

with $p^2 = 0$. Note the comparison to spinors, where $\psi(x)$ lives in spinor space, while $A_\mu(x)$ is directly a 4-vector. We define a basis of ϵ_μ^r to correspond to physical polarization states of the photon, but a photon only has two possible polarizations, and both are transverse: $\vec{\epsilon}\cdot\vec{p} = 0$, $\epsilon^\mu = (0, \vec{\epsilon})$. If we align $\vec{p}$ along $\hat{z}$, then we have the familiar left-handed & right-handed circularly polarized vectors, $\epsilon^\mu = \frac{1}{\sqrt{2}}(0, 1, \mp i, 0)$

Note that $\epsilon^\mu \epsilon_\mu = -1$.

We can also write the full Lagrangian for QED.

$$\mathcal{L} = \bar{\psi} i \not{\partial} \psi - m\bar{\psi}\psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + Qe\,\bar{\psi}\gamma^\mu\psi A_\mu .$$

We recognize the first two terms as the Dirac Lagrangian and the third term as the Lagrangian for Maxwell's eqns. The fourth term is the interaction of fermion ψ with vector field A_μ, proportional to gauge coupling e & charge Q.

This is the first Lagrangian that arises from a <u>gauge symmetry</u>, which is an internal symmetry that ~~that~~ transforms fields according to position-dependent phases. If we restrict ourselves to exponential phases parametrized by $\alpha(x)$, then the transformation of the field

$$\psi(x) \to \psi'(x) = e^{-iQ\alpha(x)}\,\psi(x)$$

where Q = charge of ψ.

Note that $\bar{\psi}\, i \not{\partial}\, \psi$ is no longer invariant:

$$\begin{aligned}
\bar{\psi}\, i\not{\partial}\,\psi \to \bar{\psi}'\, i\not{\partial}\,\psi' &= \bar{\psi}\left(e^{+iQ\alpha(x)}\right) i\gamma^\mu \partial_\mu\left(e^{-iQ\alpha(x)}\psi\right) \\
&= \bar{\psi}\, i\not{\partial}\,\psi + \bar{\psi}\, i\gamma^\mu(\partial_\mu\alpha)\cdot\psi\cdot(-iQ) \\
&= \bar{\psi}\, i\not{\partial}\,\psi + Q\,\bar{\psi}\gamma^\mu\psi\,(\partial_\mu\alpha)
\end{aligned}$$

Comparing to $Qe\bar{\psi}\gamma^\mu\psi A_\mu$, we see that the second term cancels if we transform

$$A_\mu \to A_\mu' = A_\mu - \frac{1}{e}\partial_\mu\alpha$$

Warning: We will always adopt the convention that covariant derivatives include a -1, so $D_\mu = \partial_\mu - ieQA_\mu$ (and a non-Abelian interaction is $-ig\,t^a A_\mu^a$), so that the corresponding interaction term for a fermion is $\bar{\psi}\, i\not{D}\,\psi \supset +\bar{\psi}\, eQ\not{A}\,\psi$. P+S is ~~con~~ sometimes inconsistent about this sign because they implicitly take $Q=-1$ for the electron field.

The fact that $\alpha(x)$ is dependent on x makes the symmetry transformation a <u>local</u> symmetry as opposed to a <u>global</u> symmetry. Since $\alpha(x)$ = const. is a global $U(1)$ sym., we call QED a <u>gauge $U(1)$</u>.

With the QED feynman rules, we can already present, calculate + discuss several fundamental processes that are physically relevant + measurable at tree-level. To do so, we should first remark that <u>QED is real</u> + corresponds to the QFT of any charged fermion interacting with a photon. For our purposes of studying elementary processes, we will only need to consider two separate fermion fields: electrons + muons.

Note: The electron field can excite both electrons and positrons (the charge conjugate of the electron). It is simply convention which takes the role of the particle + which is the antiparticle, but ~~they are considered~~ $\bar{\psi}$ is always implicitly counted when discussing (Dirac) fermion fields.

Given that electrons have charge -1 + mass of 511 keV, and muons have charge -1 and mass of 105 MeV, we can write the required QED Lagrangian as

$$\mathcal{L} = \bar{e}\, i\not{\partial} e - m_e \bar{e} e + \bar{\mu}\, i\not{\partial}\mu - m_\mu \bar{\mu}\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + Q_e e(\bar{e}\gamma^\mu e A_\mu) + Q_\mu e(\bar{\mu}\gamma^\mu \mu A_\mu),$$

where we have abused notation by reusing e for both the electron field and the QED gauge coupling. Here $Q = -1$ for both e + μ.

We will generally use the notation of the "covariant derivative", $D_\mu = \partial_\mu - ieQA_\mu$, which directly connects the spacetime derivative to the gauge interaction with A_μ.

It is, for example, straightforward to check that if $\psi(x) \to \psi'(x) = e^{iQ\alpha(x)}\psi(x)$, then

$$D_\mu \psi(x) \to D_\mu \psi'(x) = e^{iQ\alpha(x)} D_\mu \psi(x),$$

and so the field & its covariant derivative transform the same way.

Given the electron, muon, & photon fields, we have the following 2-to-2 processes to consider in QED:

1. $e^+e^- \to \mu^+\mu^-$ — Electrons scattering into muons
2. $e^-\mu^- \to e^-\mu^-$ — Electron-muon scattering off each other
3. $e^-\gamma \to e^-\gamma$ — Compton scattering
4. $e^+e^- \to \gamma\gamma$ — Pair annihilation to photons
5. $e^+e^- \to e^+e^-$ — Bhabha scattering
6. $e^-e^- \to e^-e^-$ — Møller scattering

We will discuss each in turn.

Nomenclature:

1. QED is a vector gauge symmetry.

$$\mathcal{L}_{int} = Q_e e (\bar{e}\gamma^\mu e A_\mu)$$
$$= Q_e e (\bar{e}\gamma^\mu (P_L + P_R) e A_\mu),$$

so LH + RH fermions experience same charge.

It is possible to charge LH & RH fields differently. This would be called a chiral symmetry, and it is central to the electroweak symmetry of the SM. But, it also requires a consistency condition known as anomaly cancellation (which we will not address in this course).

(2) The fact that we have two separate fermion fields with different masses but the same QED charges is called a flavor symmetry; flavor symmetries occur when the gauge interactions are identical for multiple copies of fermions (or multiple scalars).

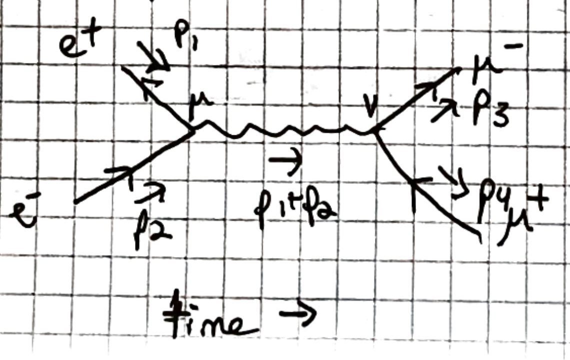
(3) The flavor symmetry guarantees that we can find a basis of fields in the mass eigenbasis that do not have flavor-changing interactions. Only following gauge symmetry, we could add

$$\mathcal{L}_{FCNC} = Q_e \left(\bar{e} \gamma^\mu \mu A_\mu + \bar{\mu} \gamma^\mu e A_\mu \right).$$

But we can then define a new combination of fields that eliminates these flavor-changing-neutral-currents (FCNCs).

(4) The fermion + scalar fields charged under a gauge symmetry are referred as the matter content, since they are typically the building blocks for what we recognize as long-lived, massive particle states.

Matrix element for $e^+e^- \to \mu^+\mu^-$.



(a) Continuous fermion flow
(b) Label vertices by Lorentz index
(c) Label lines by fermion flow + momentum flow
(d) Write $i\mathcal{M}$ by working backwards along fer. lines.

For $e^+e^- \to \mu^+\mu^-$, we have only this diagram (@ tree-level)

$$i\mathcal{M} = \bar{u}^s(p_3)\,(-ie\gamma^\nu)\,v^{s'}(p_4)\left(\frac{-ig_{\mu\nu}}{(p_1+p_2)^2+i\epsilon}\right)$$
$$\bar{v}^r(p_1)\,(-ie\gamma^\mu)\,u^{r'}(p_2)$$

$$i\mathcal{M} = \frac{ie^2\,\bar{u}^s(p_3)\,\gamma_\mu\,v^{s'}(p_4)\,\bar{v}^r(p_1)\,\gamma^\mu\,u^{r'}(p_2)}{(p_1+p_2)^2+i\epsilon}$$

For the cross section, we need to square:

$$|\mathcal{M}|^2 = \frac{e^4}{(p_1+p_2)^4}\left(\bar{u}(p_3)\,\gamma_\mu\,v(p_4)\right)\left(\bar{v}(p_1)\,\gamma^\mu\,u(p_2)\right)$$
$$\cdot\left(\bar{u}(p_2)\,\gamma^\nu\,v(p_1)\right)\left(\bar{v}(p_4)\,\gamma_\nu\,u(p_3)\right)$$

We only need to keep $+i\epsilon$ when we integrate over loops (+ need convergence of momenta at ∞).

Useful identities:

$$(\bar{u}\,\gamma_\mu\,v)^\dagger = \bar{v}\,\gamma_\mu\,u$$

$$(\bar{u}\,\gamma_\mu\gamma_5\,v)^\dagger = \bar{v}\,\gamma_\mu\gamma_5\,u$$

And same for $u \leftrightarrow v$, on both sides independently (so $(\bar{u}\,\gamma_\mu\,u)^\dagger = \bar{u}\,\gamma_\mu\,u$).

Next time: Spin sums, trace technology.