Quantum Field Theory 1.
Felix Yu
Lecture 2. Today: Spin and Poincaré / Lorentz symmetry
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Announcements: Discussion session - see email
HW 1 posted, due Monday at 10 am.
QM primer posted later today.

Last time, we started with the discussion of the scattering amplitude,

$$S_{\alpha\beta} = \langle \alpha_{out} | S | \beta_{in} \rangle$$

where α and β are asymptotic states (= particles that can be measured by detectors). We emphasized how the S-matrix (a mathematical function) is the goal of quantum field theory calculations, which itself is based on the physical principles of combining quantum mechanics, special relativity, and field theory. These will be reflected in the S-matrix as mathematical restrictions of unitarity (= no loss of total probability) and locality / causality (= spacelike amplitudes will vanish).

We start with the requirement that S is Poincaré invariant. Poincaré symmetry is the combination of Lorentz symmetry with translation invariance, where Lorentz symmetry in turn is comprised of rotational invariance and boost invariance.

The most powerful way to understand Poincaré symmetry, Lorentz symmetry, or any symmetry (for that matter) is ~~[scribble]~~ to understand the group structure. (cf. Symmetrien in Physik course)

Groups are central to a branch of mathematics known as abstract algebra. [Certainly abstract, very different than familiar algebra.]

Rules of group theory:
Groups are sets of elements with a multiplication rule.

1. If $x, y \in G$, then $x \cdot y \in G$. (Closure)
2. There is an identity, $e \cdot x = x \cdot e = x \ \forall x$ (Identity)
3. $\forall x \in G$, inverse exists & $x \cdot x^{-1} = x^{-1} \cdot x = e$ (Inverse)
4. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (Associativity)

Furthermore, if $a \cdot b = b \cdot a$ for all $a, b \in G$ (Commutativity), then the group is Abelian. Otherwise, we say the group is non-Abelian.

This is very abstract, so let's build intuition by considering some illustrative examples.

The set of real numbers w/o $\emptyset$ form a group, where the group multiplication rule is regular multiplication.

So, $x, y \in \mathbb{R}^{\emptyset}$, $x \cdot y \in \mathbb{R}$, identity is 1, inverse exists (as long as $\emptyset$ is excluded), & multiplication is associative. (It does not matter the order in which you group successive multiplications.)

We also get a group for $\mathbb{R}$ (including $\emptyset$) using the group multiplication rule of addition. Using this rule, the identity element is now $\emptyset$, A ~~differnt~~ different group is the integers $\mathbb{Z}$ under addition. Since the addition of any two integers is another integer, this satisfies closure.

A more instructive example is the rotation group $SO(3)$, which is the rotations by the Euler angles of $\hat{x}$, $\hat{y}$, and $\hat{z}$. Here, the group elements can be constructed by separate rotations about $\hat{x}$:

$$\begin{pmatrix} 1 & & \\ & \cos\theta_x & \sin\theta_x \\ & -\sin\theta_x & \cos\theta_x \end{pmatrix}$$

and rotations about $\hat{y}$ $= \begin{pmatrix} \cos\theta_y & & \sin\theta_y \\ & 1 & \\ -\sin\theta_y & & \cos\theta_y \end{pmatrix}$

and rotations about $\hat{z}$ $= \begin{pmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(All empty entries are 0).

We implicitly understand rotations in this way by their action on a given 3-vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, but group theory requires us to advance one level further into abstraction and think about the mathematical structure created by the rotations themselves. Explicitly, let us fix some rotation angles θ_x and θ_y. When we rotate a given 3-vector by θ_x then θ_y, we generally have a new rotation that involves θ_z (so θ_x rotations + θ_y rotations do not close the algebra, but θ_x, θ_y & θ_z can be used to express any arbitrary rotation).

We would write the rotated 3-vector as

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \text{seq. of} \\ \theta_x, \theta_y, \\ \theta_z \text{ rots} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

All of the structure of the group, however, is already encoded by the fact that the reference vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ was arbitrary (this is the ~~con~~ connection to more familiar algebraic equations like $f(x) = 2x+6$).
We will thus characterize the group structure by the sequence of θ_x, θ_y, and θ_z rotations without any specific reference vector. (If needed, we can always restore the reference vector to see explicitly a group element in action, but this is only a crutch + the specific choice of 3-vectors ~~known~~ and 3×3 matrices as group elements is known as a representation of the group.)

Since any rotation can be constructed by the angles θ_x, θ_y + θ_z, we can always express any group element as a sequence of θ_x, θ_y + θ_z rotations. It is useful (since the group is continuous and not discrete) to take infinitesimal rotations about $\hat{x}, \hat{y}, + \hat{z}$, and then take $\theta_x, \theta_y, \theta_z$ as multiplicative numbers that multiply the most basic rotations:

$$J_1 = i \begin{pmatrix} 0 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}, \quad J_2 = i \begin{pmatrix} 0 & & 1 \\ & 0 & \\ -1 & & 0 \end{pmatrix}$$

$$J_3 = i \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}$$

and then a rotation element is generated by $\exp(i\theta_i J_i)$. (Proper way to exponentiate: perform Taylor series expansion.) Note that rotations are not commutative (recall combinations of exps. requires Baker-Campbell-Hausdorff lemma.)

The "most basic" rotations are generators of the rotation group, and there are 3 for $SO(3)$. Note any matrix in this group has det $+1$, so this is the special, orthogonal restriction on 3×3 matrices.

Lorentz symmetry also includes boosts: we need to consider the 4-vector as our basic representation, so embed previous $J_i = i\begin{pmatrix} 0 & \\ & (3\times 3) \end{pmatrix}$ and also write infinitesimal boosts

$$K_1 = i\begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad K_2 = i\begin{pmatrix} 0 & & -1 & \\ & 0 & & \\ -1 & & 0 & \\ & & & 0 \end{pmatrix}, \quad K_3 = i\begin{pmatrix} 0 & & & -1 \\ & 0 & & \\ & & 0 & \\ -1 & & & 0 \end{pmatrix}$$

with any Lorentz transformation acting on a 4-vector as

$$\Lambda = \exp(i\theta_i J_i + i\beta_j K_j)$$

The group structure is $O(1,3)$ and preserves a "norm" (= some invariant like a distance) with metric $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$.

Having identified the generators of Lorentz symmetry, we can encapsulate the group behavior by writing down the group multiplication rule restricted to only the generators.

This set of relations is known as the algebra.
What operation ~~is~~ is needed? Commutator!

Already closed, SO(3) algebra! $\rightarrow [J_i, J_j] = i\epsilon_{ijk} J_k$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

Convention $(\epsilon_{123} = 1)$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

(Intuition: Closure guaranteed → always express RHS as linear comb. of J+K. Q: does repeated action by generator stay with same set of generators? Can also find subgroups and commuting generators.)

Finally, we can connect to spin (quantized as half-integers) by reshuffling the Lorentz generators:

$$J_i^+ = \frac{1}{2}(J_i + iK_i), \quad J_i^- = \frac{1}{2}(J_i - iK_i)$$

The algebra, using these generators, is now

$$[J_i^+, J_j^+] = i\epsilon_{ijk} J_k^+$$

$$[J_i^-, J_j^-] = i\epsilon_{ijk} J_k^-$$

$$[J_i^+, J_j^-] = 0$$

So, the J^+ and J^- generators each generate an SU(2) subgroup of O(1,3), and they do not mix (commutator is 0 between J^+ + J^-).

Projecting out the unnecessary (invariant subspace) of J_i^+ + J_i^-, we can reduce the representation from 4-vectors to 2-vectors, where each J_i^+ is identified ~~as~~ a the Pauli matrix $\frac{\sigma_i}{2} \equiv \tau_i$ (and same for J_i^-, but again the Pauli matrices act on separate 2-vectors). From QM, the 2-vectors are spin ½ representations, so we can identify $S_1 \oplus S_2$ as the representations of Lorentz symmetry, ~~Recog~~ where S_1 + S_2 are each half-integers. Recognizing that the regular angular rotations are $\vec{J} = \vec{J}^+ + \vec{J}^-$, we identify the familiar total spin for particles as $SO(3)$ representations = (direct sum of $SU(2) \oplus SU(2)$):

$SU(2) \oplus SU(2)$	$(0,0)$	$(\frac{1}{2},0)$	$(0,\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(1,0)$	$(1,1)$
$SO(3)$	0	$\frac{1}{2}$	$\frac{1}{2}$	$1 \oplus 0$	1	$2 \oplus 1 \oplus 0$

(The separate $SU(2)$ algebras are the LH + RH invariant subspaces for ~~the~~ Weyl fermions from the Dirac eqn. To see this, ~~we~~ need explicity representations + spinors, which we will do when we get to fermion fields).

All of this background justifies by we assign fields a given spin, since the Poincaré (specifically Lorentz) symmetry admits irreducible representations of half-integer conserved quantum numbers = spin.

Finally, we can also emphasize the connection between symmetries, invariance under symmetry transformations, and group theory.

When we study invariance under symmetry transformations, we assign objects to transform in a specific way (either discretely or continuously). This naturally furnishes a representation of the underlying group. Invariance comes from requiring that combinations of representations have no <u>net</u> transformation after all individual representations are applied. For continuous reps., this amounts to an independence on the continuous parameters multiplying the group generators. As an example, check that $\vec{x} \cdot \vec{y}$ is invariant of $\theta_x, \theta_y, \theta_z$ from $SO(3)$ rotations.

Next time: Scalar fields.