Lecture 16.
Quantum Field Theory 1.
Felix Yu
June 7, 2021.

Last time: Feynman rules for ϕ^4 theory, vacuum bubble exponentiation
Today: Cross sections + S-matrix, matrix elements, decay widths

The central result from last time was that we only need to consider connected, fully contracted diagrams for n-pt. correlation fcns:

$$\langle \Omega | T\{\phi(x_1)\dots\phi(x_n)\} | \Omega \rangle \equiv \text{(sum all connected, fully contracted diagrams with n external pts.)}$$

In this lecture, we will now relate correlation fcns. to S-matrix amplitudes.

The key concept here is the asymptotic state, which is a wavepacket of definite momentum constructed in the far past or detected in the far future. We can thus consider setting up an experiment with two wavepackets in the far past, and then measure particles at the far future. The corresponding probability is the amplitude squared,

$$P = |\langle \phi_1 \phi_2 \dots | \phi_A \phi_B \rangle|^2, \text{ for initial}$$

states $|\phi_A \phi_B\rangle$.

If we set $|\phi_A\rangle$ + $|\phi_B\rangle$ to have definite momenta, $\vec{p}_A$ and $\vec{p}_B$, then we can write the wavepacket as

$$|\phi_A\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_A}} \phi_A(k_A) |k_A\rangle$$

where A can be thought of as all quantum numbers that label the state, $\phi_A(\vec{k}_A)$ is the Fourier transform of the ~~spatl~~ spatial waveform centered around $\vec{p}_A$ (typically $e^{i\vec{k}\cdot\vec{x}}$) and $|\vec{k}_A\rangle = \sqrt{2E_k}\, a_k^\dagger |0\rangle$ in the free theory.

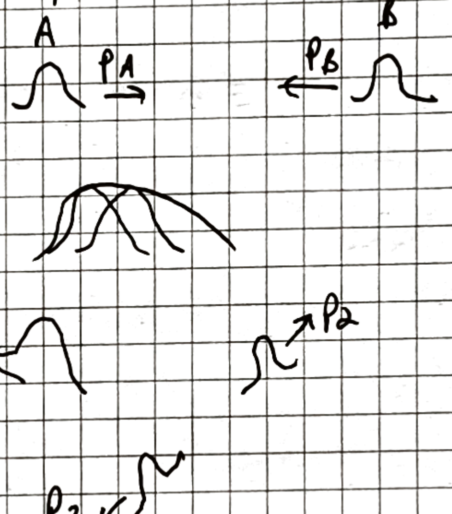
Note if $\int \frac{d^3k}{(2\pi)^3} |\phi(k)|^2 = 1$, then $\langle \phi | \phi \rangle = 1$

We will generally focus on initial states that are localized individually in momentum space but broad enough to not overlap in position space.

Cartoon: Initial states

Interaction

final states

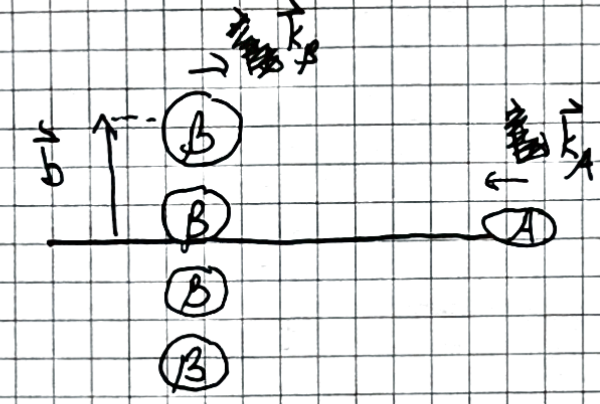


Note that both well before ~~or~~ well after the interaction, we can consider the states as essentially free, except for the complication of the shift in vacuum energy from the interacting theory $\Rightarrow$ hence we take $T \to \pm\infty$ and call them asymptotic states.

We now have the corresponding "in" state as

$$|\phi_A \phi_B\rangle_{in} = \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{1}{\sqrt{2E_A}} \frac{1}{\sqrt{2E_B}} \phi_A(\vec{k}_A)\, \phi_B(\vec{k}_B)\, e^{-i\vec{b}\cdot\vec{k}_B} |\vec{k}_A \vec{k}_B\rangle_{in}$$

where $\vec{b}$ is the transverse impact factor.



Given that the out states are particle detectors that resolve energy + momenta but do not resolve spatial distances at the level of de Broglie wavelengths, we can label the out states by $_{out}\langle \vec{p}_1 \vec{p}_2 \ldots |$ the definite momenta.

Note that different observers see equivalent but not identical states. For observer $\mathcal{O}$, if the clock is set to $t=0$, and for a second observer $\mathcal{O}'$ sets the clock to $t'=0$ at $t=\tau$, then $\mathcal{O}$ sees the system in a state $|\alpha\rangle$ while $\mathcal{O}'$ sees the system in the state $|\alpha'\rangle = \exp(-iH\tau)|\alpha\rangle$. So, to relate far future + far past observed states, we need the factor $\lim_{T\to\infty(1-i\epsilon)} \exp[-iH\,2T]$ to transform states to same basis of state vectors (Weinberg, p. 109).

Thus, $$_{out}\langle \vec{p}_1 \vec{p}_2 \ldots | \vec{k}_A \vec{k}_B \rangle_{in} = \lim_{T\to\infty(1-i\epsilon)} \langle \vec{p}_1 \vec{p}_2 \ldots | e^{-iH(2T)} | \vec{k}_A \vec{k}_B \rangle$$

where the states in RHS are all at a given common reference time. (We work in Heisenberg picture)

We can now define the S-matrix (as in lecture 1)

$$_{out}\langle \vec{p}_1 \vec{p}_2 \ldots | \vec{k}_A \vec{k}_B \rangle_{in} = \langle \vec{p}_1 \vec{p}_2 \ldots | S | \vec{k}_A \vec{k}_B \rangle$$

since the interactions perform a unitary operation on the set of asymptotic states, $S = \lim_{T\to\infty(1-i\epsilon)} U(T,-T)$

We can conventionally extract the identity from S,

$S = \mathbb{1} + iT$, where we recognize that the scattering matrix S has a piece where states do not interact: $S \ni \mathbb{1}$, and a piece where states actually transition/change, $S \ni iT$.

Translational invariance implies that the transition matrix can be written as

$$\langle \vec{p}_1 \vec{p}_2 \ldots | \, iT \, | \vec{k}_A \vec{k}_B \rangle = (2\pi)^4 \, \delta^{(4)}(k_A + k_B - p_1 - p_2 - \ldots) \; i\mathcal{M}(k_A, k_B \to p_1, p_2, \ldots)$$

where the δ-fcn. enforces energy-momentum conservation.

(This also reduces the Poincaré algebra to Lorentz algebra.)

Now, all particles are on the mass shell, $p_i^0 \equiv E_{\vec{p}_i} = \sqrt{|\vec{p}_i|^2 + m_i^2}$

particles = asymptotic states

The quantity $\mathcal{M}$ is called the <u>scattering amplitude</u>.

We now have $\mathcal{M}$ that encodes the selection rules from the interaction Hamiltonian, + separates these dynamics from the pure kinematics. We can now relate $\mathcal{M}$ to a scattering cross section: The probability for initial state $|\phi_A \phi_B\rangle$ to scatter into a final state of n particles with momenta $\vec{p}_i$ inside intervals of size d^3p_i is

$$P(AB \to 12\ldots) = \left(\prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}} \right) \left| {}_{\text{out}}\langle \vec{p}_1 \ldots \vec{p}_n | \phi_A \phi_B \rangle_{\text{in}} \right|^2$$

For single target particle A + many incident target particles B_i with different impact params. $\vec{b}$, the number of scattering events is

$$N = \sum_{\substack{\text{incident} \\ \text{particles} \\ B_i}} P_i = \int d^2 b \; n_B \, P(\vec{b})$$

n_B: # of B particles per unit area (assume constant)

This gives a differential cross section

$$d\sigma = \frac{N}{n_B N_A} = \int d^2b\, P(\vec{b})$$

$N_A = 1$

which is differential in final-state momenta. We now combine everything:

$$d\sigma = \int d^2b \left(\prod_{i=1}^{n} \frac{d^3p_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right)$$

$$\cdot \int \frac{d^3k_A}{(2\pi)^3} \int \frac{d^3k_B}{(2\pi)^3} \frac{\phi(\vec{k}_A)\phi(\vec{k}_B)}{(2E_{k_A} 2E_{k_B})^{1/2}} \int \frac{d^3k_A'}{(2\pi)^3} \int \frac{d^3k_B'}{(2\pi)^3} \frac{\phi_A^*(\vec{k}_A')\phi^*(\vec{k}_B')}{\sqrt{(2E_{k_A'} 2E_{k_B'})}}$$

$$e^{i\vec{b}\cdot(\vec{k}_B'-\vec{k}_B)} (2\pi)^8 \delta^{(4)}(k_A+k_B-p_1-\dots-p_n)\delta^{(4)}(k_A'+k_B'-p_1-\dots-p_n)$$

$$\cdot \mathcal{M}(k_A,k_B \to p_1\dots p_n)\, \mathcal{M}^*(k_A', k_B' \to p_1 \dots p_n)$$

We use $\int d^2b\, e^{i\vec{b}\cdot(\vec{k}_B'-\vec{k}_B)} = (2\pi)^2 \delta^{(2)}(\vec{k}_{B\perp}'-\vec{k}_{B\perp})$

and combine with $\delta^{(4)}(k_A'+k_B'-p_1'-\dots-p_n')$ to perform all six integrations over $d^3k_A'\ d^3k_B'$.

$$\Rightarrow \int d^2k_{B\perp}' \Rightarrow \vec{k}_{B\perp}' = \vec{k}_{B\perp}$$

$$\int d^2k_{A\perp}' \Rightarrow \vec{k}_{A\perp}' = -\vec{k}_{B\perp}' + \sum_{i=1}^{n} \vec{p}_{i\perp} = \vec{k}_{A\perp}$$

$$\int dk_A'^z \int dk_B'^z\, \delta(k_A'^z + k_B'^z - \sum_i p_i^z)\, \delta(E_A' + E_B' - \sum_i E_i)$$

$$= \int dk_A'^z\, \delta\left(\sqrt{(k_A'^z)^2 + m_A^2} + \sqrt{(k_B'^z)^2 + m_B^2} - \sum_i E_i\right)\Bigg|_{k_B'^z = -k_A'^z + \sum_i p_i^z}$$

$$= \frac{1}{\left|\frac{k_A'^z}{E_A'} - \frac{k_B'^z}{E_B'}\right|}\Bigg|$$ evaluated with $k_A'^z + k_B'^z = \sum_i p_i^z = k_A^z + k_B^z$ and $E_A' + E_B' = \sum_i E_i = E_A + E_B$, $\vec{k}_{i\perp}' = \vec{k}_{i\perp}$

$$\equiv \frac{1}{|v_A - v_B|}$$ is the relative velocities of the beams as viewed in the lab frame.

$$= \frac{1}{\left|\frac{k_A^2}{E_A} - \frac{k_B^2}{E_B}\right|}$$

We simplify to

$$d\sigma = \left(\prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}}\right) \int \frac{d^3 k_A}{(2\pi)^3} \int \frac{d^3 k_B}{(2\pi)^3} \frac{|\phi_A(\vec{k}_A)|^2 |\phi_B(\vec{k}_B)|^2}{2E_A 2E_B}$$

$$\cdot \frac{E_A E_B}{|k_A^2 E_B - k_B^2 E_A|} \cdot (2\pi)^4 \cdot \delta^{(4)}(k_A + k_B - p_1 - p_2 \ldots - p_n)$$

$$|\mathcal{M}(k_A, k_B \to p_1 \ldots p_n)|^2$$

The wave packets have done their job of regularizing the squared δ-fcn. Given that realistic detectors cannot resolve small variations from spread of wavepackets ϕ_A + ϕ_B, we replace $k_A \to p_A$ + $k_B \to p_B$ in $\delta^{(4)}(\ldots)$ distribution and matrix element + integrate over wavepackets:

$$d\sigma = \left(\prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}}\right) (2\pi)^4 \delta^{(4)}(p_A + p_B - p_1 - \ldots - p_n)$$

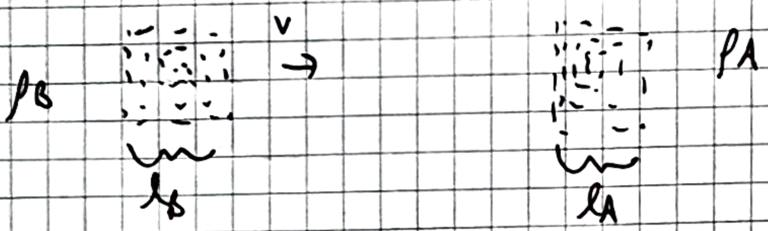
$$\cdot \underbrace{\frac{1}{4} \frac{1}{|p_A^2 E_B - p_B^2 E_A|}}_{\text{flux of incident particles}} \qquad \underbrace{|\mathcal{M}(p_A, p_B \to p_1, \ldots, p_n)|^2}_{\text{squared transition amplitude}}$$

The $$\left(\prod_i^n \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{p_i}}\right) (2\pi)^4 \delta^{(4)}(p_A + p_B - \Sigma p_i)$$

is the n-particle phase space factor.

Note $$\frac{1}{4(p_A^2 E_B - p_B^2 E_A)} = \frac{1}{2E_A 2E_B} \frac{1}{|v_A - v_B|}.$$

To visualize the cross section, take target A +
hit it with bunch of particles B.



ρ = number density per unit volume

frame-independent $\quad \sigma \equiv \dfrac{\text{number of scattering events}}{\rho_A \, l_A \, \rho_B \, l_B \, A}$

A = cross sectional area common to two bunches.

For small enough beams, approximate ρ_A + ρ_B as constant.

$$\# \text{ of events} = \sigma \, l_A \, l_B \int d^2x \; \rho_A(x) \rho_B(x)$$
$$= \sigma \frac{N_A N_B}{A}$$

N_A + N_B are total numbers of A + B particles

Practical ~~[illegible]~~ examples:

- proton beams on proton beams (LHC)
- proton beam on target
- electron beam on positron beam
- electron beam on proton beam
- cosmic rays

Can also discuss width:

$$\Gamma \equiv \frac{\text{number of decays per unit time}}{\text{number of parent particles present}}$$

Next time: Flux factor, kinematics, phase space