Lecture 10.
Quantum Field Theory 1.
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Last time: Quantization & anticommutation relations for Dirac field. Spin-statistics theorem.

Today: Free quantized theory
Spin 1/2 of fermion field
~~Discrete symmetries~~

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We have Dirac free quantized fields in 4-component notation:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( a_p^s u^s(p) e^{-ip\cdot x} + b_p^{s\dagger} v^s(p) e^{ip\cdot x} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( b_p^s \bar{v}^s(p) e^{-ip\cdot x} + a_p^{s\dagger} \bar{u}^s(p) e^{ip\cdot x} \right)$$

The ladder operators obey anticommutation quantization conditions:

$$\{ a_p^r, a_q^{s\dagger} \} = \{ b_p^r, b_q^{s\dagger} \} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \delta^{rs}$$

$$\& \quad \{ a_p^r, a_q^s \} = \{ a_p^{r\dagger}, a_q^{s\dagger} \} = \{ b_p^r, b_q^s \} = \{ b_p^{r\dagger}, b_q^{s\dagger} \} = 0$$

Then, the equal-time commutations of fields $\psi$ & $\psi^\dagger$ are

$$\{ \psi_a(\vec{x},t), \psi_b^\dagger(\vec{y},t) \} = \delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}$$

$$\{ \psi_a(\vec{x}), \psi_b(\vec{y}) \} = \{ \psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y}) \} = 0$$

Annihilation operators define vacuum state $|0\rangle$:

$$a_p^s |0\rangle = b_p^s |0\rangle = 0$$

Performing the usual Legendre transform of the Lagrangian, we get the Hamiltonian

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} \left( a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s - (2\pi)^3 \delta^{(3)}(0) \right)$$

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Note that the zero-pt. energy is negative here (and infinite). This leads to the possibility that a theory that relates bosons + fermions has vanishing zero pt. energy, which turns out is a requirement in supersymmetric theories.

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Total momentum operator is

$$\vec{P} = \int d^3x \; \psi^\dagger (-i\vec{\nabla}) \psi = \int \frac{d^3p}{(2\pi)^3} \sum_s \vec{p} \left( a_p^{s\dagger} a_p^s + b_p^{s\dagger} b_p^s \right)$$

$\Rightarrow$ $a_p^{s\dagger}$ creates fermion with energy $E_p$ + momentum $\vec{p}$.

$b_p^{s\dagger}$ creates antifermion with energy $E_p$ + momentum $\vec{p}$.

Normalization of states:

$$|\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}} \; a_p^{s\dagger} |0\rangle$$

$$\Rightarrow \langle \vec{p}, r | \vec{q}, s \rangle = 2E_p (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \delta^{rs}$$

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Angular momentum + spin.

The total angular momentum operator is

$$\vec{J} = \int d^3x \; \psi^\dagger \left( \vec{x} \times (-i\vec{\nabla}) + \frac{1}{2} \vec{\sigma} \right) \psi$$

Justification: angular momentum is $\vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$

We have fermions, $\psi(x) \to \psi'(x) = \Lambda_{1/2} \psi(\Lambda^{-1}x)$

At a fixed point,

$$\delta\psi(x) = \psi'(x) - \psi(x) = \Lambda_{1/2} \psi(\Lambda^{-1}x) - \psi(x)$$

Using $\Lambda_{1/2} = \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right)$ ③

$$\simeq 1 - \frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}$$

we can study an infinitesimal rotation about $\hat{z}$-axis.

Then $\omega_{12} = -\omega_{21} = \theta$, all other params. are zero.

$$\Lambda_{1/2} \simeq 1 - \frac{i}{2}\omega_{12} S^{12} + \frac{i\omega_{21}}{2} S^{21}$$

$$\simeq 1 - \frac{i\theta}{2} S^{12} + \frac{i\theta}{2} S^{21}$$

Recall $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix} = \frac{1}{2}\epsilon^{ijk}\Sigma^k$

$$\Lambda_{1/2} \simeq 1 - \frac{i\theta}{2}\left(\frac{1}{2}\epsilon^{12k}\Sigma^k - \frac{1}{2}\epsilon^{21k}\Sigma^k\right)$$

$$= 1 - \frac{i\theta}{2}\left(\epsilon^{123}\Sigma^3\right) = 1 - \frac{i\theta}{2}\Sigma^3$$

Using $\epsilon^{123} = 1$ and antisymmetry of indices.

$$\delta\psi(x) = \left(1 - \frac{i}{2}\theta\Sigma^3\right)\psi(t, x+\theta y, y-\theta x, z) - \psi(x)$$

$$= -\theta\left(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3\right)\psi(x)$$

We get the time component of the conserved Noether current is

$$j^0 = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\Delta\psi = -i\bar{\psi}\gamma^0\left(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3\right)\psi$$

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For relativistic fermions, total angular momentum is not straightforward to decompose into ladder operators.

Easiest to choose rest frame & consider $J_z$.

Want $J_z a_0^{s\dagger}|0\rangle = \pm\frac{1}{2} a_0^{s\dagger}|0\rangle$, where $\pm\frac{1}{2}$ is the eigenvalue for spin.

Easiest argument: (~~S~~Schwartz, p.175)

$$\Lambda_{1/2}(\theta=2\pi) = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

so if we have state $|\psi_1\psi_2\rangle = |\psi_\uparrow(\vec{x})\psi_\uparrow(-\vec{x})\rangle$,
then rotation by $\pi$ gives $\Lambda_{1/2}(\pi) = i$ for each spinor.
$|\psi_1\psi_2\rangle \to -|\psi_2\psi_1\rangle$ when particles are interchanged.
Moreover, $\Lambda_{1/2}(\theta_z) = \exp(i\theta_z S_{12}) = \begin{pmatrix} \exp(\frac{1}{2}\theta_z) & & & \\ & \exp(-\frac{1}{2}\theta_z) & & \\ & & e^{(\frac{1}{2}\theta_z)} & \\ & & & e^{(-\frac{1}{2}\theta_z)} \end{pmatrix}$

Eigenvalues are $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, + -\frac{1}{2}$, and they operate on basis spinors $\xi^s = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ and $\eta^s = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ for the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ irrepr. of Lorentz group.

We can conclude that

$$J_z\, a_0^{s\dagger}|0\rangle = \pm\frac{1}{2}a_0^{s\dagger}|0\rangle$$

$$J_z\, b_0^{s\dagger}|0\rangle = \mp\frac{1}{2}b_0^{s\dagger}|0\rangle$$

(opposite order in $J_z$, get neg. sign)

Upper signs for $\xi^s = \begin{pmatrix}1\\0\end{pmatrix} + \eta^s = \begin{pmatrix}1\\0\end{pmatrix}$,
lower signs for $\xi^s = \begin{pmatrix}0\\1\end{pmatrix} + \eta^s = \begin{pmatrix}0\\1\end{pmatrix}$.

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Dirac propagator:

We can also calculate the propagators,

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle$$

$$= \int\frac{d^3p}{(2\pi)^3}\cdot\frac{1}{2E_p}\sum u_a^s(p)\bar{u}_b^s(p)\,e^{-ip\cdot(x-y)}$$

$$= (i\partial\!\!\!/_x + m)_{ab}\int\frac{d^3p}{(2\pi)^3}\cdot\frac{1}{2E_p}e^{-ip\cdot(x-y)}$$

$$\langle 0|\bar{\Psi}_b(y)\,\Psi_a(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\sum_s v^s_a(p)\,\bar{v}^s_b(p)\,e^{-ip\cdot(y-x)}$$
$$= -(i\not{\partial}_x + m)_{ab}\int\frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,e^{-ip\cdot(y-x)}$$

We again have the retarded Green's function

$$S_R^{ab}(x-y) \equiv \theta(x^0-y^0)\,\langle 0|\{\Psi_a(x),\bar{\Psi}_b(y)\}|0\rangle$$
$$= (i\not{\partial}_x + m)\,D_R(x-y).$$

The $S_R$ is a Green's fcn. of the Dirac operator,

$$(i\not{\partial}_x - m)\,S_R(x-y) = i\delta^{(4)}(x-y)\,\mathbb{1}_{4\times 4}.$$

We will generally use the Feynman prescription $+i\epsilon$:

$$S_F(x-y) = \int\frac{d^4p}{(2\pi)^4}\,\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)}$$
$$= \begin{cases} \langle 0|\Psi(x)\bar{\Psi}(y)|0\rangle & \text{for } x^0>y^0 \ \text{(close below)} \\ -\langle 0|\bar{\Psi}(y)\Psi(x)|0\rangle & \text{for } x^0<y^0 \ \text{(close above)} \end{cases}$$
$$\equiv \langle 0|T\,\Psi(x)\bar{\Psi}(y)|0\rangle$$

The Fourier transform is most commonly used in evaluating Feynman diagrams;

$$\tilde{S}_F(p) = \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

; note this is matrix-valued in spinor space. $(m \equiv m\,\mathbb{1}_{4\times4})$

Next time: discrete symmetries.