08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie Quantum Field Theory I

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Homework set 2

Due May 2, 2024 by start of discussion session.
Please note how long it took you to solve each problem.

2-1, 40 pts. We start with the free field solution for the real Klein-Gordon field,

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right) , \tag{1}$$

and the equal-time commutation relations

$$[\phi(x), \phi(y)]|_{x^0=y^0} = 0 , \tag{2}$$

$$[\pi(x), \pi(y)]|_{x^0=y^0} = 0 , \tag{3}$$

$$[\phi(x), \pi(y)]|_{x^0=y^0} = i\delta^{(3)}(\vec{x} - \vec{y}) . \tag{4}$$

A, 15 pts. Explicitly verify that the Hamiltonian

$$H = \int d^3x \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right] \tag{5}$$

becomes

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(a^\dagger_{\vec{p}}\, a_{\vec{p}} + \frac{1}{2}(2\pi)^3 \delta^{(3)}(0) \right) . \tag{6}$$

B, 15 pts. Explicitly verify that the total momentum operator

$$\vec{P} = -\int d^3x\ \pi(\vec{x})\ \vec{\nabla}\phi(\vec{x}) \tag{7}$$

can be reexpressed as

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}}\, a_{\vec{p}} . \tag{8}$$

C, 10 pts. Finally, write a paragraph explaining the significance of the $\delta^{(3)}(0)$ term from part A, making analogy to the quantum mechanical harmonic oscillator as necessary. Does this term have physical consequences?

D, Extra credit, 10 pts. For completeness, we can adopt a regularization procedure for the infinite zero-point energy from part C by considering the energy density. (The simplest approach is to confine the spectrum to a finite box of volume V; for any box size, the 3-dimensional delta function grows with size V, so dividing by V is equivalent to setting the 3D delta function to 1.) In this case, the zero point energy density is

$$\epsilon_0 = \frac{E_0}{V} = \int d^3p \frac{1}{2} E_{\vec{p}} \, . \tag{9}$$

Calculate the energy density over all momentum modes $|\vec{p}|$. You should get an *ultraviolet divergence* from the upper limits of integration of $|\vec{p}|$ at ∞, corresponding to the assumption that the Hamiltonian is valid to arbitrarily high energy scales. To address the divergence, we can replace the upper limit of integration by Λ_{UV}: what is the new expression for the energy density? (In the end, to extract an observable quantity, we should remember that any scalar potential can include a finite constant contribution $V(\phi) \supset V_0 + \frac{1}{2}m^2\phi^2 + \ldots$, which we need to keep for this discussion. The sum of the regularized zero-point energy density from the KG Hamiltonian and the V_0 term then comprise the *dark energy* cosmological constant, which is measured to be $\epsilon_0 \simeq (10^{-3}\text{ eV})^4$.)

2-2, 60 pts. Consider a real function $f(x)$, which has a unique global minimum at $x = 0$. We define the integral

$$I(\alpha) = \int dx \exp\left\{-\frac{1}{\alpha} f(x)\right\} \, . \tag{10}$$

A, 20 pts. Method of steepest descent. Perform a Taylor expansion of $f(x)$ in the exponent and show that for $\alpha \to 0$, the integral has the asymptotic expansion

$$I(\alpha) = e^{-f_0/\alpha} \sqrt{\frac{2\pi\alpha}{f_0^{(2)}}} \left\{ 1 + \left(\frac{5}{24} \frac{\left(f_0^{(3)}\right)^2}{\left(f_0^{(2)}\right)^3} - \frac{3}{24} \frac{f_0^{(4)}}{\left(f_0^{(2)}\right)^2} \right) \alpha + \mathcal{O}(\alpha^2) \right\} \, . \tag{11}$$

Here $f_0 = f(0)$ and $f_0^{(n)}$ is the n-th derivative evaluated at $x = 0$. You will find the Gaussian integral $\int dx\ x^n e^{-ax^2}$ useful.

B, 20 pts. Method of stationary phase. Repeat the derivation in part A with the replacement of $\alpha \to i\alpha$. Can you justify why the expansions in part A and part B are both valid?

C, 20 pts. Causality violation in relativistic quantum mechanics. One of the main motivations for quantum field theory is that relativistic quantum mechanics allows for non-vanishing amplitudes for non-causal particle propagation. The amplitude for a free particle to propagate from $\vec{x}_0$ to $\vec{x}$ in quantum mechanics in $U(t) =$

$\langle \vec{x}|e^{-iHt}|\vec{x}_0\rangle$. Using the relativistic expression for energy, $H = E = \sqrt{|\vec{p}|^2 + m^2}$, verify that

$$U(t) = \frac{1}{2\pi^2|\vec{x}-\vec{x}_0|}\int\limits_0^\infty dp\; p\sin\left(p|\vec{x}-\vec{x}_0|\right)e^{-it\sqrt{p^2+m^2}}\;, \tag{12}$$

with the usual $p = |\vec{p}|$ for the magnitude of the momentum vector. Using the method of stationary phase, obtain the leading term in the asymptotic expansion of $U(t)$ for spacelike separation of $|\vec{x}|^2 \gg t^2$, and explain how this term violates casuality.