08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie Quantum Field Theory I

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Homework set 1

Due April 24, 2024, by start of lecture
Please note how long it took you to solve each problem

1-1, 15 pts. Practice with $\hbar = c = 1$. Throughout this course, we will set $\hbar = c = 1$. Then, following $E = mc^2$, mass dimensions are equivalent to energy dimensions. Consequently, following the photon energy formula, $E = \frac{2\pi\hbar c}{\lambda}$, length dimensions are equivalent to inverse energy dimensions. (You can develop intuition for these "natural units" [where $\hbar = c = 1$] by taking a given length scale and thinking of the necessary photon energy needed to probe such a scale.) (A) The Large Hadron Collider is the highest center-of-mass energy collider in the world, operating at $\sqrt{s} = 14$ TeV. What equivalent length scale is probed by the collider? (B) The Cosmic Microwave Background has a temperature of $T_\gamma = 2.7255$ K. What is the equivalent photon energy and length scale? (C) In some disciplines of high energy physics, it is common to also set the Newton constant G and the Boltzmann constant k_B each to 1. Using this convention, construct the "Planck length" as the dimensionally correct combination of $\hbar$, G, and c (you will not need k_B). What is one Planck length in SI units?

1-2, 35 pts. (Peskin and Schroeder, problem 2.1A) Consider the action for classical electromagnetism without source terms,

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) , \tag{1}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \, . \tag{2}$$

Derive Maxwell's equations as the Euler-Lagrange equations from this action, using the components of $A_\mu(x)$ as the dynamical variables. You should recover the familiar form of Maxwell's equations by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

1-3, 25 pts. In your own words, write a paragraph to explain how this choice of commutation relations quantizes the scalar field:

$$[\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \tag{3}$$

$$[\phi(\vec{x}), \phi(\vec{y})] = 0 \tag{4}$$

$$[\pi(\vec{x}), \pi(\vec{y})] = 0 \, . \tag{5}$$

If needed, make reference to analogous operators from quantum mechanics.

1-4, 25 pts. We can define the Fourier transform of a function $f(x)$ to be

$$\tilde{f}(k) = \int dx f(x) e^{-ikx} \ . \tag{6}$$

(A) Evaluate the Fourier transform of the Heaviside function $\theta(x)$. (Introduce a regulator in the exponent by replacing e^{-ikx} with $e^{-(ik+\epsilon)x}$ that helps the integral to converge at infinity, and then take the limit for $\epsilon \to 0^+$. *Hint*: You should be able to express the answer using the delta function and principal value.) (B) What is the Fourier transform for $f(x) = 1/(x^2 + a^2)$ for some real constant a? You should close the integration contour in the upper or lower complex x-plane.