Lecture 9.
Quantum Field Theory 1.
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May 10, 2021

Last time: Free fermion field, Dirac solutions, fermion bilinears, vector + axial-vector currents

Today: Quantized Dirac fermion field.

Recall that we constructed free field solutions for the Dirac Lagrangian $\psi(x) = \int \frac{d^3p}{(2\pi)^3} u_s(p) e^{-ip\cdot x}$

and $\psi(x) = \int \frac{d^3p}{(2\pi)^3} v_s(p) e^{ip\cdot x}$, where $u_s(p)$ + $v_s(p)$ are 4-component spinors built out of the basis spinors: $u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_s \\ \sqrt{p\cdot\bar\sigma}\,\xi_s \end{pmatrix}$, $v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_s \\ -\sqrt{p\cdot\bar\sigma}\,\eta_s \end{pmatrix}$

where $\xi_1 = \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi_2 = \eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are constant 2-component basis spinors (spin up + spin down).

We will interpret $u^s(p)$ + $v^s(p)$ as particles (pos. freq. solns) + anti-particles (neg. freq. solns.), respectively, valid solns to the Dirac eqn. with arbitrary momentum.

In analogy with complex scalar fields, we can now quantize the Dirac free field. The main difficulty or complication will be the question of whether to take commutation or anti-commutation relations for our ladder operators.

(See Section 12.3 of Schwartz)

Recall,
$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{E_p}} \left(a_p e^{-ip\cdot x} + b_p^{\dagger} e^{ip\cdot x} \right)$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{E_p}} \left(a_p^{\dagger} e^{ip\cdot x} + b_p e^{-ip\cdot x} \right)$$

Analogously, we now associate momentum mode ladder operators for our spinor field free solutions:

$$\Psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p^s u^s(p) e^{-ip\cdot x} + b_p^{s\dagger} v^s(p) e^{+ip\cdot x} \right)$$

$$\overline{\Psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p^{s\dagger} \bar{u}^s(p) e^{-ip\cdot x} + b_p^s \bar{v}^s(p) e^{ip\cdot x} \right)$$

$$E_p = \sqrt{|\vec{p}|^2 + m^2}$$

As usual, $\Psi(x)$ annihilates incoming particles & creates antiparticles, $\overline{\Psi}(x)$ annihilatates incoming anti-particles & creates particles.

We have two options for a_p^s & b_p^s, $a_p^{s\dagger}$ & $b_p^{s\dagger}$:

(Bosonic) commutation relations:

$$[a_p^s, a_q^{s'\dagger}] = [b_p^s, b_q^{s'\dagger}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\delta_{ss'}$$

(Fermionic) anti-commutation relations.

$$\{a_p^s, a_q^{s'\dagger}\} = \{b_p^s, b_q^{s'\dagger}\} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\delta_{ss'}$$

Making the wrong choice will cause many problems for a Lorentz-invariant theory: 1 Breakdown of Lorentz invariance and negative norm states, 2 violation of stability, 3 violation of causality.

We will sketch the justifications for each briefly, but we can already highlight the spin-statistics theorem (Pauli, 1940):

The requirements of Lorentz invariance, positive energies & positive norms. require that particles with integer spins have commutation relations and particles with half-integer spins have anti-commutation relations. Equivalently, identical bosonic particles are symmetric under interchange, while identical fermionic particles are anti-symmetric under interchange.

Brief argument:

$$|\ldots s_1 \vec{p}_1 n \ldots s_2 \vec{p}_2 n \ldots\rangle = \sqrt{2E_1}\, a^\dagger_{\vec{p}_1 s_1 n} \cdots \sqrt{2E_2}\, a^\dagger_{\vec{p}_2 s_2 n} \ldots |0\rangle$$

additional quantum numbers that match

These multiparticle states are normalized:

$$\langle s_1 \vec{p}_1 n_1 \ldots | s_1' \vec{p}_1' n_1' \ldots\rangle = \prod_i \delta_{n_i n_i'}\, \delta_{s_i s_i'}\, 2E_i\, (2\pi)^3\, \delta^{(3)}(\vec{p}_i - \vec{p}_i')$$

Now, create multiparticle state by adopting different order:

$$|\ldots s_2 \vec{p}_2 n \ldots s_1 \vec{p}_1 n \ldots\rangle = \ldots \sqrt{2E_2}\, a^\dagger_{\vec{p}_2 s_2 n} \ldots \sqrt{2E_1}\, a^\dagger_{\vec{p}_1 s_1 n} \ldots |0\rangle$$

Given they are identical, these must be the same physical state & can only differ by normalization.

We have fixed the magnitude of the normalization, so they can only differ by a phase:

$$|\ldots s_1 \vec{p}_1 n \ldots s_2 \vec{p}_2 n \ldots\rangle = \underset{\alpha}{\underset{\|}{e^{i\phi}}}\, |\ldots s_2 \vec{p}_2 n \ldots s_1 \vec{p}_1 n \ldots\rangle$$

We cannot have α depend on the Lorentz algebra quantum numbers s & $\vec{p}$, since we have constructed all possible reps. of these states and there are no remaining one-dimensional phase reps. So α can at most depend on the internal quantum numbers n. (There are no non-trivial paths to characterize the exchange of the particles in 3+1 dimensions, since the covering space is simply connected. Not true in 2+1 $\Rightarrow$ leads to anyons.)

We can swap back, $\Rightarrow$ $\alpha_n^2 = +1 \Rightarrow \alpha_n = \pm 1$.

So $\alpha_n = +1 \Rightarrow$ particles are bosons,

$$a^\dagger_{\vec{p}_1 s_1 n}\, a^\dagger_{\vec{p}_2 s_2 n} |\Psi\rangle = a^\dagger_{\vec{p}_2 s_2 n}\, a^\dagger_{\vec{p}_1 s_1 n} |\Psi\rangle$$

for all $|\Psi\rangle$, &

Commutation: $[a^\dagger_{\vec{p}_1 s_1 n}, a^\dagger_{\vec{p}_2 s_2 n}] = [a_{\vec{p}_1 s_1 n}, a_{\vec{p}_2 s_2 n}] = 0$

By normalization $\langle \vec{p}_1 | \vec{p}_2 \rangle = (2E_1)(2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$, $[a_{p_1}, a^\dagger_{p_2}] = (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$

For $\alpha_n = -1$, $\Rightarrow$ particles are fermions,

$$\{a^\dagger_{\vec{p}_1 s_1 n}, a^\dagger_{\vec{p}_2 s_2 n}\} = \{a_{\vec{p}_1 s_1 n}, a_{\vec{p}_2 s_2 n}\} = 0$$

and $$\{a^\dagger_{\vec{p}_1 s_1 n}, a_{\vec{p}_2 s_2 n}\} = (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)\, \delta_{s_1 s_2}.$$

This already demonstrates the key relationship between spin and statistics & multiparticle states and commutation / anti-commutation relations.

Let us briefly observe consequences of using the wrong relations for scalar & fermion fields.

Lorentz invariance:

$$\langle 0| T\{\phi^*(x)\phi(0)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{ip\cdot x}$$

Lorentz-invariant time-ordered product, commutation relations imposed.

For anti-commutations, need anti-time ordering

$$T\{\psi(x)\chi(y)\} = \psi(x)\chi(y)\,\theta(x_0-y_0) - \chi(y)\psi(x)\,\theta(y_0-x_0)$$

Then $\langle 0| T\{\phi^*(x)\phi(0)\}|0\rangle$

$$= \int \frac{d^4p}{(2\pi)^4} \cdot \frac{E}{\sqrt{|\vec{p}|^2+m^2}} \frac{-i}{p^2-m^2+i\epsilon} e^{ip\cdot x}$$

integration var. (E); E_p ($\sqrt{|\vec{p}|^2+m^2}$)

& this is not Lorentz invariant anymore.

For spinors, we will derive the feynman propagator:

$$\langle 0| T\{\psi(0)\bar{\psi}(x)\}|0\rangle = (-i\not{\partial}+m)\int \frac{d^4p}{(2\pi)^4} \frac{i}{E^2-E_p^2+i\epsilon} e^{ip\cdot x}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{ip\cdot x} \quad \text{is L.I.}$$

Stability:

Recall for scalars:

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \left[a_p^\dagger a_p + b_p b_p^\dagger \right]$$

Using commutation relations,

$$= \int \frac{d^3p}{(2\pi)^3} E_p \left[a_p^\dagger a_p + b_p^\dagger b_p + \text{ZPE} \right]$$

relating to number operators,

$$\Delta E = \int \frac{d^3p}{(2\pi)^3} E_p \left[\Delta\#\text{particles} + \Delta\#\text{antiparticles} \right]$$

So states with more particles + antiparticles have more energy.

If we had used anti-commutation for scalars,

$$\Delta E = \int \frac{d^3p}{(2\pi)^3} E_p \left[\Delta\#\text{particles} - \Delta\#\text{antiparticles} \right],$$

and then the energy is lowered by increasing the number of particles. So <u>stability</u> is gone.

Similarly, for fermions,

$$E = \sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left(a_p^{\dagger s} a_p^s - b_p^s b_p^{s\dagger} \right)$$

$$\underset{\text{anticommutators}}{\Rightarrow} \quad E = \sum_s \int \frac{d^3p}{(2\pi)^3} E_p \left(a_p^{\dagger s} a_p^s + b_p^{s\dagger} b_p^s - ZPE \right)$$

The same argument about particles + antiparticles applies if we used commutation relations for fermions (which switches $+b^\dagger b$ to $-b^\dagger b$).

Causality

We have already seen that commutation relations for bosons ~~causes~~ enforces causality for spacelike separation. In the same way, anticommutation relations will enforce causality for spacelike separtion of fermion fields. (See P+S, p. 56)

Summary: Quantization of fermions requires anti-commutation relations.

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u^s(p) e^{-ip\cdot x} + b_p^{\dagger s} v^s(p) e^{ip\cdot x} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(b_p^s \bar{v}^s(p) e^{-ip\cdot x} + a_p^{s\dagger} \bar{u}^s(p) e^{ip\cdot x} \right)$$

$$\{ a_p^r, a_q^{s\dagger} \} = \{ b_p^r, b_q^{s\dagger} \} = (2\pi)^3 \delta^{(3)} (\vec{p}-\vec{q}) \delta^{rs}$$

$\Rightarrow$ Equal time ~~con~~ anticommutation relations.

$$\{ \psi_a(\vec{x}), \psi_b^\dagger(\vec{y}) \} = \delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}$$

$$\{ \psi_a(\vec{x}), \psi_b(\vec{y}) \} = \{ \psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y}) \} = 0.$$

& $\quad a_{\vec{p}}^s |0\rangle = b_{\vec{p}}^s |0\rangle = 0$

annihilate vacuum.