Lecture 8.

Quantum Field Theory 1.

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Last time: Dirac matrices + Dirac spinors + Dirac Lagrangian

Today: Free fermion theory.

Last time, we found the Dirac Lagrangian + Dirac equation:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi \quad , \quad \not{\partial} \equiv \gamma^\mu \partial_\mu \quad , \quad \bar{\Psi} \equiv \Psi^\dagger \gamma^0$$

$$\Rightarrow \begin{cases} (i\not{\partial} - m)\Psi = 0 \\ -i(\partial_\mu \bar{\Psi})\gamma^\mu - m\bar{\Psi} = 0 \end{cases}$$

We also found that solutions for the Dirac equation also implied they were solutions of the Klein-Gordon equation.

So, we can build $\Psi(x)$ out of plane-wave solutions,

$$\Psi(x) = u(p)\, e^{-ip\cdot x} \quad , \quad \text{for} \quad p^2 = m^2 ,$$

and where we require $\Psi(x)$ to satisfy the Dirac equation $\Rightarrow$ $u(p)$ satisfies the (Fourier transform) Dirac equation.

$$(i\not{\partial} - m)\, u(p)\, e^{-ip\cdot x} = i(-ip_\mu \gamma^\mu)\, e^{-ip\cdot x}\, u(p) - e^{-ip\cdot x}\, u(p)\, m = 0$$

$$= (\gamma^\mu p_\mu - m)\, u(p) = 0$$

Since Dirac equation is boost (+ Lorentz) invariant, construct solutions in rest frame, with $p^\mu = (m, \vec{0})$, then boost to other frames to construct arbitrary $u(p)$.

Then, $(\gamma^\mu p_\mu - m)\,u(p) = (m\gamma^0 - m\mathbb{1})\,u(p)$

$$= m\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p) = 0$$

remember that these are 2x2 blocks.

$\Rightarrow \quad u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$, for ξ a two-component spinor.

We include $\sqrt{m}$ as a normalization factor, and we require $\xi^\dagger \xi = 1$. We can recognize ξ as the usual two-component spinor + then $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a spin "up".

Note we have constructed the 4-component spinor but we have two independent spin degrees of freedom. Thus, after quantization, the particles associated with $\psi(x)$ are fermions with 2 degrees of freedom, parametrized by the ξ spinor.

We can construct $u(p)$ by boosting from rest frame to new frame (see derivation on p. 45-46 of Peskin + Schroeder); the result is

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$$

, which is actually valid for any direction of $\vec{p}$.

$\sqrt{p\cdot\sigma}$ = + square root of eigenvalue.

$= \dfrac{p\cdot\sigma + m}{\sqrt{2(p_0+m)}}$, which avoids square roots of matrices.

Note that $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, under large boosts, sends

$$u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \rightarrow \sqrt{2E}\begin{pmatrix} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

, reducing to pure lower component Weyl spinor. For $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \rightarrow \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \end{pmatrix}$$

For large boosts, $E = \sqrt{m^2 + |\vec{p}|^2}$, $E \sim |\vec{p}|$.

This demonstrates the connection between helicity and chirality. As ~~a~~ boost approaches c, helicity approaches chirality. Recall that chirality is the intrinsic spin representation of the Lorentz group, while helicity is the projection of the spin vector on the momentum.

Lorentz transformations of ψ & $\bar{\psi}$ transfer to $u(p)$ & $\bar{u}(p) = u^{\dagger}\gamma^0$.

The normalization condition of $\bar{u}u = 2m\xi^{\dagger}\xi = 2m$.

When we identify spin index on ξ, we have

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}, \quad s = 1, 2.$$

$\Rightarrow$ $\bar{u}^r(p)u^s(p) = 2m\delta^{rs}$. These are positive frequency solutions to the Dirac equation, $\psi(x) = u(p)e^{-ip\cdot x}$, $p^2 = m^2$, $p^0 > 0$.

We can also construct negative frequency solutions, $p^0 < 0$,

$$\psi(x) = v(p)\,e^{+ip\cdot x}, \quad p^2 = m^2, \quad p^0 < 0.$$

Here, $v(p)$ is again two linearly independent spinor solutions.

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}, \quad s = 1, 2.$$

Normalization: $\bar{v}^r(p)v^s(p) = -2m\delta^{rs}$.

We also have $\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0$.

Aside: We will generally want to calculate sums over all possible spins, since we generally consider unpolarized spin states. As such, we can also determine the completeness relations for spin ½ states.

Then, instead of $\bar{u}^r(p)\, u^s(p)$, we want an open spin tensor,

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \cdot \left(\xi^{s\dagger}\sqrt{p\cdot\bar\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\sigma}\right)$$

since $\sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{pmatrix}$$

Note $(p\cdot\sigma)(p\cdot\bar\sigma) = p^2 = m^2$.

$$= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix}$$

$$\Rightarrow \sum_s u^s(p)\,\bar{u}^s(p) = \gamma\cdot p + m = \not{p} + m.$$

Similarly, $\sum_s v^s(p)\,\bar{v}^s(p) = \not{p} - m.$

Lorentz covariance and fermion bilinears.

We have constructed $\bar\psi\psi$ as a Lorentz vector and $\bar\psi\psi$ as a Lorentz 4-vector, but we can now saturate the complete set of 4×4 matrix structures $\bar\psi\Gamma\psi$. There are 16 such matrices.

$\bar\psi\psi$: 1 $\mathbb{1}$ matrix

$\bar\psi\gamma^\mu\psi$: 4 γ^μ matrices

$\bar\psi\gamma^{\mu\nu}\psi = \bar\psi\left(\frac{1}{2}[\gamma^\mu,\gamma^\nu]\right)\psi = \bar\psi\gamma^{[\mu}\gamma^{\nu]} = \bar\psi(-i\sigma^{\mu\nu})\psi$: 6 tensors

$\bar\psi\gamma^{\mu\nu\rho}\psi = \bar\psi\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho]}\psi$ = 4 matrices $= \bar\psi\gamma^\mu\gamma^5\psi$

$\bar\psi\gamma^{\mu\nu\rho\sigma}\psi = \bar\psi\gamma^{[\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma]}\psi$ = 1 matrix $= \bar\psi\gamma^5\psi$.

Can check that each $\gamma^{\mu\nu}$, $\gamma^{\mu\nu\rho}$, $\gamma^{\mu\nu\rho\sigma}$ transforms as successively higher Lorentz covariant tensor.

The $\gamma^{[\mu}\gamma^{\nu]}$ notation indicates antisymmetrization over the bracketed indices.

Clarifying note: $\gamma^{\mu\nu\rho}$ and $\gamma^{\mu\nu\rho\sigma}$ are sums over all permutations of the Lorentz indices, divided by the appropriate $n!$ counting the number of permutations, and with a (-1) factor for every pairwise swap of indices.

$$\text{So}\quad \gamma^{\mu\nu\rho} = \frac{1}{6}\Big(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\mu\gamma^\rho\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\rho + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\rho\gamma^\nu\gamma^\mu + \gamma^\rho\gamma^\mu\gamma^\nu \Big)$$

and similarly for $\gamma^{\mu\nu\rho\sigma}$.

To streamline notation, we use the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$, with the convention that

$$\epsilon^{0123} = -\epsilon_{0123} = +1.$$

← Note upper + lower differ in sign!

We have also simplified the last two bilinears using

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{-i}{4!}\epsilon^{\mu\nu\rho\sigma}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)$$

So, $\gamma^{\mu\nu\rho} = +i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$

and $\gamma^{\mu\nu\rho\sigma} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5$.

We now identify
$\bar\psi\psi$ = scalar
$\bar\psi\gamma^\mu\psi$ = vector
$\bar\psi\sigma^{\mu\nu}\psi$ = tensor
$\bar\psi\gamma^\mu\gamma^5\psi$ = pseudovector
$\bar\psi\gamma^5\psi$ = pseudoscalar.

The "pseudo" refers to ~~that~~ the fact that Lorentz transformations are carried by its vector ~~/~~ or scalar nature, but we also need a discrete parity transformation.

There are several nice properties about γ^5.

$$(\gamma^5)^\dagger = \gamma^5$$

$$(\gamma^5)^2 = \mathbb{1}_{4\times 4}$$

$$\{\gamma^\mu, \gamma^5\} = 0.$$

Then, $[\gamma^5, S^{\mu\nu}] = 0$, and so Dirac spinors are reducible representations according to the eigenvalue under γ^5: we have

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$ (remember, 2x2 blocks)

We see that the Weyl spinors are the irreducible representation with LH spinors with eigenvalue -1 and RH spinors as $+1$.

Chiral currents & vector currents.

We can now discuss the concept of chiral currents, which are centrally important to the construction of the Standard Model.

First, define the standard vector and axial-vector currents:

vector: $j^\mu = \bar{\psi}\gamma^\mu \psi$

axial-vector $j^{\mu 5} = \bar{\psi}\gamma^\mu \gamma^5 \psi$

We can calculate the divergences of these:

$$\partial_\mu j^\mu = (\partial_\mu \bar{\psi})\gamma^\mu \psi + \bar{\psi}\gamma^\mu(\partial_\mu \psi)$$

$$= +im\bar{\psi}\psi + \bar{\psi}(-im\psi) = 0.$$ Conserved

$$\partial_\mu j^{\mu 5} = (\partial_\mu \bar{\Psi})\gamma^\mu \gamma^5 \psi + \bar{\psi}\gamma^\mu \gamma^5 (\partial_\mu \psi)$$
$$= 2im\,\bar{\psi}\gamma^5 \psi.$$

If $m=0$, this is classically conserved.

Aside: In fact, the axial current is not generally conserved for $m=0$, because quantum effects (only evident at 1-loop order) are incompatible with the 4-dimensional nature of the γ^5 matrix. This is the beginning of the study of chiral anomalies, a very deep + rich subject out of the scope of this course.

We can reshuffle γ^μ and $\gamma^\mu\gamma^5$ into the left-handed + right-handed currents as

$$\gamma^\mu \left(\frac{1-\gamma_5}{2} \right) = \gamma^\mu P_L,$$
$$\gamma^\mu \left(\frac{1+\gamma_5}{2} \right) = \gamma^\mu P_R.$$

Then $\gamma^\mu (P_L + P_R) = \gamma^\mu$, $\quad \gamma^\mu \gamma^5 = \gamma^\mu(-P_L + P_R)$.

For $m=0$, then current densities of LH particles and RH particles are separately conserved.

Note: a necessary condition to gauge a current is that it must be conserved, so we can interpret the gauge (vector) boson as carrying charge.

Next time: quantization of fermion fields