Lecture 7.

Quantum Field Theory I.

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Last time: Fermions - Weyl fermions + 2-component spinors

Today: Dirac fermions and 4-component spinors.

At the end of the last lecture, we saw that

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right], \text{ where } \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}_{n\times n},$$

furnished a representation of the Lorentz group algebra, with γ^μ being an $n\times n$ matrix. (We need 4 such matrices that anti-commute to be $2g^{\mu\nu}$.)

Clarifying note: need 4 matrices since Lorentz index in γ^μ runs from 0 to 3. But the $n\times n$ dimension of the matrix is for n-dimensional representations, which is unrelated to the dimensions of spacetime.

What is the explicit form of these γ^μ matrices?

We cannot simply use 2-dim. reps, because we need a 4th Pauli matrix that anticommutes with the other 3.

We thus take the convention known as the <u>Weyl</u> or <u>chiral</u> representation, with

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

and each element is itself a 2×2 block.

What do the Lorentz generators look like with this representation? And ~~wh~~ what objects do these γ^μ matrices act on?

We have $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, so we can check that

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{i}{4}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

boost generators

$$= \frac{i}{4}\left(\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} - \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \right)$$

$$= \frac{-i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

and $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{i}{4}\left(\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \right)$

rotation generators

$$= \frac{i}{4}\left(\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} - \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix} \right)$$

$$= \frac{-i}{4} \begin{pmatrix} 2i\epsilon^{ijk}\sigma^k & 0 \\ 0 & 2i\epsilon^{ijk}\sigma^k \end{pmatrix}$$

$$= \frac{1}{2}\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

The 4-component state that these 4x4 matrices act on is known as a <u>Dirac spinor</u>. We can immediately identify that the Dirac spinor is nothing more than two 2-component spinors, $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, where ψ_L and ψ_R are 2-component Weyl spinors from before.

To see this, we recall that Lorentz transformations are built by exponentiation of generators with arbitrary parameters. Here, we have

$\Lambda_{1/2} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$, where $\omega_{\mu\nu}$ are the free parameters that will drop out of any Lorentz invariant transformation. $\omega_{\mu\nu}$ is an antisymmetric tensor that gives the infinitesimal angles.

For example, if we perform a Lorentz transformation on a vector, we use the vector representation of the Lorentz generators that act on 4-vectors,

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha).$$

Then, a 4-vector $x^\rho \to \exp\left(\frac{-i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)^\rho{}_\sigma x^\sigma$, where $\omega_{\mu\nu}$ are the boost + rotation angles (antisymmetric tensor params.) + ρ + σ on the RHS denote the entries of the matrix from the ~~expot~~ exponential.

We can now check how γ^μ transforms under Lorentz. Calculate the commutator with $S^{\rho\sigma}$:

$$[\gamma^\mu, S^{\rho\sigma}] = \gamma^\mu S^{\rho\sigma} - S^{\rho\sigma}\gamma^\mu$$

$$= \frac{i}{4}\left(\gamma^\mu[\gamma^\rho,\gamma^\sigma] - [\gamma^\rho,\gamma^\sigma]\gamma^\mu\right)$$

Use $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$

$$= \frac{i}{4}\left(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\sigma\gamma^\rho - \gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\rho\gamma^\mu\right)$$

$$= \frac{i}{4}\left(\cancel{2g^{\rho\sigma}}\gamma^\mu - 2\gamma^\mu\gamma^\sigma\gamma^\rho - \cancel{2g^{\rho\sigma}\gamma^\mu} + 2\gamma^\sigma\gamma^\rho\gamma^\mu\right)$$

$$= \frac{i}{4}(2)\left(-2g^{\mu\sigma}\gamma^\rho + \cancel{\gamma^\sigma\gamma^\mu\gamma^\rho} + 2g^{\mu\rho}\gamma^\sigma - \cancel{\gamma^\sigma\gamma^\mu\gamma^\rho}\right)$$

$$= i\left(-g^{\mu\sigma}\gamma^\rho + g^{\mu\rho}\gamma^\sigma\right)$$

$$= i\left(g^{\mu\rho}\delta^\sigma{}_\nu\gamma^\nu - g^{\mu\sigma}\delta^\rho{}_\nu\gamma^\nu\right)$$

$$= i\left(\mathcal{J}^{\rho\sigma}\right)^\mu{}_\nu\gamma^\nu$$

So, γ^μ transforms exactly as a regular 4-vector!

Given that γ^μ transforms as a 4-vector, we can now consider what field equation to apply to the 4-component Dirac spinor $\Psi(x)$.

Candidate 1: Klein-Gordon eqn. $(\partial^2 + m^2)\Psi = 0$.

This is true but not enough. We can actually use the fact that spinors transform as ½ reps. of Lorentz:

$$\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu,$$

$$\Lambda_{1/2} = \exp\left(-\frac{i}{2} \omega_{\mu\nu} S^{\mu\nu}\right),$$

$$\Psi \to \Lambda_{1/2} \Psi.$$

Candidate 2: Dirac equation.

$$(i\gamma^\mu \partial_\mu - m)\Psi(x) = 0.$$

This is the basic field equation for 4-component spinors. It corresponds to wave equations for spin ½ particles that obey anti-commutation relations.

First, check Lorentz invariance:

$$\begin{aligned}
(i\gamma^\mu \partial_\mu - m)\Psi(x) &\xrightarrow{\text{spacetime trans.}} (i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m)\Lambda_{1/2}\Psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}\Lambda_{1/2}^{-1}(i\gamma^\mu (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m)\Lambda_{1/2}\Psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}(i\Lambda_{1/2}^{-1}\gamma^\mu \Lambda_{1/2}(\Lambda^{-1})^\nu{}_\mu \partial_\nu - m)\Psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}(i\Lambda^\mu{}_\sigma \gamma^\sigma (\Lambda^{-1})^\nu{}_\mu \partial_\nu - m)\Psi(\Lambda^{-1}x) \\
&= \Lambda_{1/2}(i\gamma^\nu \partial_\nu - m)\Psi(\Lambda^{-1}x) \\
&= 0.
\end{aligned}$$

We can also see that the Dirac equation is the "square root" of the Klein-Gordon equation:

$$0 = (-i\gamma^\mu \partial_\mu - m)(i\gamma^\nu \partial_\nu - m)\psi$$

$$= (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi$$

$$= \left(\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu + m^2\right)\psi$$

$$= (g^{\mu\nu}\partial_\mu \partial_\nu + m^2)\psi$$

$$= (\partial^2 + m^2)\psi$$

How do we get Lorentz-invariant contractions of 4-component Dirac spinors?

First option: $\psi^\dagger \psi \to \psi^\dagger \Lambda_{1/2}^\dagger \Lambda_{1/2} \psi$

but $\Lambda_{1/2}$ is <u>not</u> Hermitian (since group is noncompact) and $\Lambda_{1/2}^\dagger \Lambda_{1/2} \neq 1$.

Solution: $\bar{\psi}\psi$, with $\bar{\psi} \equiv \psi^\dagger \gamma^0$.

$$\bar{\psi} \to \psi^\dagger \left(1 + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right)\gamma^0$$

For rotation terms, $(\mu,\nu) \to (i,j)$ + γ^0 commutes with $(S^{\mu\nu})^\dagger = S^{\mu\nu}$.

For boost terms, $(\mu,\nu) \to (0,j)$ or $(i,0)$ & γ^0 anticommutes with $S^{\mu\nu}$, and $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$, so passing γ^0 to the left removes the extra negative sign.

$$\bar{\psi} \to \psi^\dagger \left(1 + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right)\gamma^0$$

$$= \psi^\dagger \gamma^0 \left(1 + \frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right)$$

$= \bar{\psi}\Lambda_{1/2}^{-1}$, which is the desired Lorentz transformation rule.

$\Rightarrow \bar{\psi}\psi$ is Lorentz invariant.

The Dirac Lagrangian is

$$\mathcal{L}_{Dirac} = \bar{\Psi}\,(i\gamma^\mu \partial_\mu - m)\,\Psi,$$

or, using Feynman-slash notation, $\gamma^\mu \partial_\mu \equiv \not\partial$,

and in general, $\gamma^\mu x_\mu = \not x$.

arb. 4-vector x_μ

The Euler-Lagrange equations for Ψ + $\bar{\Psi}$ are then

$$(i\not\partial - m)\,\Psi = 0$$

$$-i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0$$

Connecting back to Weyl spinors + spinor space:

We identify the upper + lower 2-components of the Dirac spinor as Weyl spinors: $\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix}$.

These transform under rotations and boosts as before (Ψ_L + Ψ_R have opposite boost angles.)

Note that $\sigma^2 \Psi_L^*$ transforms as a right-handed spinor. (Complex conjugate reverses rotation angles, and

$\sigma^2 (\vec{\sigma})^* = -(\vec{\sigma})\,\sigma^2$ does the Hermitian reversal.)

It is instructive to study the Dirac equation on Weyl spinors:

$$(i\gamma^\mu \partial_\mu - m)\,\Psi$$

$$= \begin{cases} i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\,\Psi_R - m\Psi_L = 0 \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\,\Psi_L - m\Psi_R = 0 \end{cases}$$

For $m=0$, these solutions separate + we can treat Ψ_L + Ψ_R independently. So L + R denote <u>chirality</u>, the spin property that is intrinsic to the spinor for $m = 0$.

If we define $\sigma^\mu = (1, \vec{\sigma})$ + $\bar{\sigma}^\mu = (1, -\vec{\sigma})$
then we have

$$\begin{cases} (i\sigma\cdot\partial)\psi_R - m\psi_L = 0 \\ (i\bar{\sigma}\cdot\partial)\psi_L - m\psi_R = 0 \end{cases}$$

+ the Weyl equations (setting $m=0$) obtained by are

$$\begin{cases} i\vec{\bar{\sigma}}\cdot\partial\psi_L = 0 \\ i\sigma\cdot\partial\psi_R = 0 \end{cases}$$

Next time: Free particle solutions + quantization.