Lecture 5. ①

Quantum Field Theory I.

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Last time: Noether's theorem, time-dependence

Today: Antiparticles + time dependence.

Last time, we discussed the commutator $[\phi(x), \phi(y)]$:

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)$$

$$= D(x-y) - D(y-x)$$

We also saw that the commutator vanished for spacelike intervals and was nonzero for timelike intervals, which was a necessary condition for causality.

It is more instructive to consider the question of causality when we can distinguish particle and antiparticle. We can do this easily when a charge distinguishes the two, ~~as~~ as can be seen for a complex scalar field.

With complex scalar fields, the negative frequency modes are associated with antiparticles, and we must introduce a new set of creation and annihilation operators: $b^\dagger_{\vec{p}}$ and $b_{\vec{p}}$.

Remark: $\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot x} a_{\vec{p}} \Big|_{p^0 = E_{\vec{p}}}$

is wrong + fails causality (although commutation relations do hold for ϕ + ϕ^*).

So, we write

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(e^{-ip\cdot x} a_{\vec{p}} + e^{ip\cdot x} b^\dagger_{\vec{p}}\right)$$

$$; \; p^0 = E_{\vec{p}}$$

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left(e^{-ip\cdot x} b_{\vec{p}} + e^{ip\cdot x} a^\dagger_{\vec{p}}\right)$$

$$\pi(x) = \dot{\phi}^*(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left(-iE_{\vec{p}}\right)\left(e^{-ip\cdot x} b_{\vec{p}} - e^{+ip\cdot x} a^\dagger_{\vec{p}}\right)$$

$$\pi^*(x) = \dot{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left(-iE_{\vec{p}}\right)\left(e^{-ip\cdot x} a_{\vec{p}} - e^{ip\cdot x} b^\dagger_{\vec{p}}\right)$$

To ensure the canonical commutation relations, we treat a + b as their own ladder operators:

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) = [b_p, b^\dagger_{\vec{q}}]$$

All other commutators vanish: $[a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0$

$[b_{\vec{p}}, b_{\vec{q}}] = [b^\dagger_{\vec{p}}, b^\dagger_{\vec{q}}] = 0$

Mixed commutators also vanish!

Can derive free Hamiltonian and momentum operator:

$$H_{free} = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}} + \text{zero pt. energy}\right)$$

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}}\right)$$

Interpretation:

Recall that $\phi(x)$ and $\phi^*(x)$ transform oppositely under a $U(1)$ phase transformation: if $\phi(x) \to e^{i\alpha}\phi(x)$, then $\phi^*(x) \to e^{-i\alpha}\phi^*(x)$. Correspondingly, we must interpret $b^\dagger_{\vec{p}}$ and $a_{\vec{p}}$ consistently with regards to charge flow.

Since $\phi(x)|0\rangle$ creates a particle at position $\vec{x}$, we exercise the $b^\dagger_{\vec{p}}$ operator and further specify that $\phi(x)|0\rangle$ creates a + charge particle at position $\vec{x}$ via $b^\dagger_{\vec{p}}$ or annihilates a − charge particle at position $\vec{x}$ via $a_{\vec{p}}$. Similarly, $\phi^*(\vec{x})|0\rangle$ creates a − charge particle at position $\vec{x}$ with $a^\dagger_{\vec{p}}$ or annihilates a + charge particle at position $\vec{x}$ with $b_{\vec{p}}$.

So, $b^\dagger_{\vec{p}}$ + $b_{\vec{p}}$ create + annihilate + particles and $a^\dagger_{\vec{p}}$ + $a_{\vec{p}}$ create + annihilate the − antiparticles.

Then, if we consider now $[\phi(x), \phi^\dagger(y)] = D(x-y) - D(y-x)$, the first term $\phi(x)\phi^\dagger(y) = D(x-y)$ is the creation of a − charge at y that travels to x and is annihilated; the second term $\phi^\dagger(y)\phi(x) = D(y-x)$ is the creation of a + charge at x that travels to y and is annihilated. To be self-consistent, the particle + antiparticle must both exist, have equal masses, and have opposite charges. The real scalar field has no charge, so the antiparticles + particles are equivalent.

Klein-Gordon propagator

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right)$$

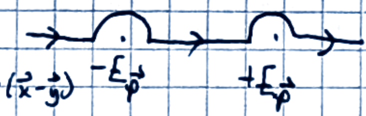
Change to d^4p integral by considering integrand as result of dp^0 integral.

$$= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}\Big|_{p^0=E_{\vec{p}}} + \frac{1}{-2E_{\vec{p}}} e^{-ip\cdot(x-y)}\Big|_{p^0=-E_{\vec{p}}} \right\}$$

$$\underset{x^0>y^0}{=} \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{(2\pi i)} \frac{-1}{p^2-m^2} e^{-ip\cdot(x-y)}$$

Check: p^2-m^2 has poles for

$p^0 = E_{\vec{p}} = \pm\sqrt{|\vec{p}|^2+m^2} \Rightarrow$



For $x^0>y^0$, need $e^{-ip^0(x-y)}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}$

we get 2 residues as above ← to vanish exponentially fast, so we take p^0 in the lower half-plane. For $x^0<y^0$, then we close the p^0 integral in the upper half-plane + get 0.

Putting these together, we need a Heaviside fcn. on (x^0-y^0):

$$D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle$$

This is the <u>retarded</u> <u>Green's</u> <u>function</u> of the Klein-Gordon operator. Vanishes for $x^0<y^0$.

Check:

$$(\underset{\partial_\mu\partial^\mu}{\partial^2} + m^2)\, D_R(x-y) = [\partial^2\theta(x^0-y^0)]\langle 0|[\phi(x),\phi(y)]|0\rangle + \theta(x^0-y^0)(\partial^2+m^2)\langle 0|[\phi(x),\phi(y)]|0\rangle + 2[\partial_\mu\theta(x^0-y^0)](\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle)$$

$$= [\partial_t\,\delta(x^0-y^0)]\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$\phi(x), \phi(y)$ are solutions of KG → $+\,0$

$$+\,2\,\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

Integrate by parts in first line:

$$= -\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$
$$+\,2\,\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$
$$= \delta(x^0-y^0)(-i\,\delta^{(3)}(\vec{x}-\vec{y})) = -i\,\delta^{(4)}(x-y)$$

We see that $D_R(x-y)$ is a <u>Green's fcn.</u> of the Klein-Gordon operator.

Recall: Green's fcns. are solutions to differential eqns. that give a δ-fcn. response. In particular, they are the building blocks of solving inhomogeneous differential eqns, since we can generally decompose sources into δ-fcn. pieces, each solved by its own Green's fcn, then superposing the solutions together.

Alternate derivation: Fourier transform to momentum space.

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \tilde{D}_R(p)$$

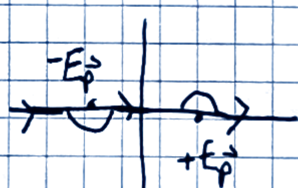
Then $\tilde{D}_R(p)$ is the solution to

$$(-p^2+m^2)\,\tilde{D}_R(p) = -i$$

$$\Rightarrow \tilde{D}_R(p) = \frac{i}{p^2-m^2}$$

$$\Rightarrow D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2} e^{-ip\cdot(x-y)}$$

Feynman prescription: Enclose poles with $+i\epsilon$:



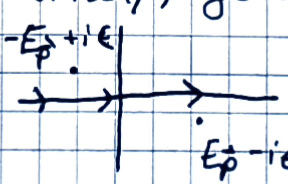
Modify denominator to

$$p^2-m^2 \to p^2-m^2+i\epsilon$$

Then, poles are $(p^0)^2 = |\vec{p}|^2+m^2-i\epsilon$

$$\Rightarrow p^0 = E_{\vec{p}} - i\epsilon \quad \text{and} \quad p^0 = -E_{\vec{p}} + i\epsilon$$

Pictorially, get contour as



Will then take $\epsilon \to 0$ at end.

For $x^0 > y^0$, we close the contour below and get $D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}$

For $x^0 < y^0$, we close the contour above and get

$$D(y-x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{ip\cdot(x-y)}$$

Together, this defines

$$D_F(x-y) = \begin{cases} D(x-y), & x^0 > y^0 \\ D(y-x), & x^0 < y^0 \end{cases}$$

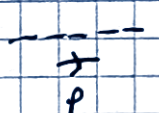
$$= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(x)\phi(y)|0\rangle$$

$$\equiv \langle 0|T\phi(x)\phi(y)|0\rangle$$

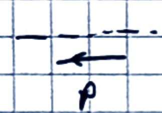
with T = time-ordering, which rearranges the following operators to put the latest time at the left-most position + earliest all the way to the right. $D_F(x-y)$ = Feynman propagator + reflects the propagation amplitude.

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

This will correspond to a momentum-space Feynman rule for a scalar propagator,



$$\frac{i}{p^2-m^2+i\epsilon}$$ or



(since $D_F(x-y) = D_F(y-x)$.)