Lecture 26.

Quantum Field Theory 1.

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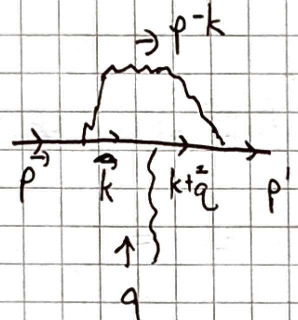
Last time: Electron vertex calculation at one-loop

Today: Finish electron vertex calc. Perturbation Lagrangian.



$$p+q=p' \quad , \quad \Delta \equiv -xy\,q^2 + (z-1)^2 m^2$$

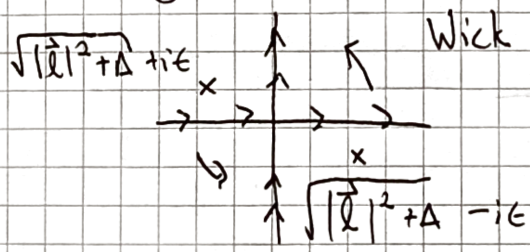
We left off with the matrix element loop calculation.

★ Was calculated in 4D. Will redo in d-dimensions.

$$i\mathcal{M} = 4e^3\,\epsilon_\mu(q)\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\int \frac{d^4\ell}{(2\pi)^4}\,\frac{\bar{u}(p')\Big(-\tfrac{1}{2}\ell^2\gamma^\mu + (1-y)(1-x)\gamma^\mu q^2 + (1-4z+z^2)m^2\gamma^\mu + zm(z-1)\,i\sigma^{\mu\nu}q_\nu\Big)u(p)}{(\ell^2-\Delta+i\epsilon)^3}$$

Now, in the complex momenta plane, the denominator for $(\ell^0)^2 - |\vec{\ell}|^2 - \Delta + i\epsilon$ has poles at

$\ell^0 = \pm\sqrt{|\vec{\ell}|^2+\Delta} \mp i\epsilon$, allowing us to modify the contour of integration along the purely imaginary axis without crossing either pole.



Wick rotation: We deform the ℓ^0 integration by performing a Wick rotation replacing $\ell^0 = i\ell^0_E$, $\vec{\ell} = \vec{\ell}_E$.

In this way, we change the integration in Minkowski space to an integration in Euclidean space, allowing a more intuitive d-dimensional volume integral in "normal" space.

Divergences are then regularized using dimensional regularization. Inverted by 't Hooft & Veltman (1972), dim. reg. is the main regularization procedure used in QFT today. Alternatives include hard momentum cutoffs & Pauli-Villars, but we will focus on dim. reg.

There are two key aspects to dim. regularization & regularization in general.

1. Regularization is necessary because our Feynman rules (as formulated) lead us to indefinite integrals, and so we need a regularization prescription to help us define the physical / principal value of such integrals when the infinities are cancelled.

2. In dim. reg., we calculate integrals in d-dimensions. Since the loop integrals always involve propagators over unknown loop momenta, we can guarantee that the integral <u>converges</u> and is well-defined for small enough d. Then, by the completeness of the Γ-fcn, which only has isolated poles at $\Gamma(z)$ for $z = 0$, & negative integers $= z$, we can always separate divergences into poles of the Γ-fcn & otherwise analytic expressions. The poles, which could be either IR or UV divergences, are then seen as $\frac{1}{\epsilon}$ singularities, with $d = 4 - \epsilon, \epsilon \to 0$. Any physical observable will cancel $\frac{1}{\epsilon}$ poles.

For the electron vertex function, we have two loop integrals with differing powers of ℓ^2 in the numerator. For each, we change to a d-dimensional Euclidean space integral by Wick-rotation, then (using the identity derived in HW 12) we write the corresponding expressions using Γ-fcns:

We derived

$$\int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^n} = \overbrace{i\int \frac{d^d \ell_E}{(2\pi)^d} \frac{(-1)^n}{(\ell_E^2+\Delta)^n}}^{\text{intermediate change to Euclidean space}}$$

$$= \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n-d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-d/2}$$

And $$\int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

Remark: It is commonly needed to extract the behavior near

$$\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}} \left(\frac{1}{\Delta}\right)^{2-d/2} \to \frac{1}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta + O(\epsilon)\right)$$

for $\epsilon = 4-d$.

Remark: $\gamma^\mu[\gamma\ldots]\gamma_\mu$ identities are modified in d-dim. See P+S (A.55.).

Remark: The $i\epsilon$ in the denominator of propagators (where ϵ here is unrelated to the dim. reg. ϵ) is then a prescription in $\log\Delta$ to choose the correct branch cut. Will be necessary whenever loop particles can go on-shell (+ is directly connected to optical theorem).

For the electron vtx fcn., we redo the numerator calculation in d-dimensions.

New identities: $\gamma^\mu \gamma_\mu = d$

$$\gamma^\mu \gamma^\alpha \gamma_\mu = -(2-\epsilon)\gamma^\alpha = -(d-2)\gamma^\alpha$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 4g^{\alpha\beta} - (4-d)\gamma^\alpha\gamma^\beta$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\delta \gamma_\alpha = -2\gamma^\delta\gamma^\beta\gamma^\alpha + (4-d)\gamma^\alpha\gamma^\beta\gamma^\delta$$

For $d = 4-\epsilon$, we have

$$\gamma^\mu \gamma^\alpha \gamma_\mu = -(2-\epsilon)\gamma^\alpha$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 4g^{\alpha\beta} - \epsilon\gamma^\alpha\gamma^\beta$$

$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\delta \gamma_\mu = -2\gamma^\delta\gamma^\beta\gamma^\alpha + \epsilon\gamma^\alpha\gamma^\beta\gamma^\delta$$

$$i\mathcal{M} = -e^3 \epsilon_\mu(q) \int \frac{d^4k}{(2\pi)^4} \bar{u}(p')\gamma^\nu \frac{(\not{k}'+m)\gamma^\mu(\not{k}+m)\gamma_\nu}{(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)((p-k)^2+i\epsilon)} u(p)$$

Numerator simplifications: $\quad k' \equiv k+q$

Use $\ell \equiv k + xq - zp$

Will also need an IR regulator for photon:

change $(p-k)^2 + i\epsilon \to (p-k)^2 + \mu^2 + i\epsilon$ with μ^2 as fictitious photon mass.

$$\gamma^\nu(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma_\nu$$

$$= \gamma^\nu(\not{\ell}-x\not{q}+z\not{p}+\not{q}+m)\gamma^\mu(\not{\ell}-x\not{q}+z\not{p}+m)\gamma_\nu$$

Drop linear terms in ℓ

$$= \gamma^\nu\big(\not{\ell}\gamma^\mu\not{\ell} + ((1-x)\not{q}+z\not{p})\gamma^\mu(-x\not{q}+z\not{p})$$
$$+ ((1-x)\not{q}+z\not{p})\gamma^\mu m + m\gamma^\mu(-x\not{q}+z\not{p})$$
$$+ m^2\gamma^\mu\big)\gamma_\nu$$

$$= -2\,\not{\ell}\gamma^\mu\not{\ell} + \epsilon\,\not{\ell}\gamma^\mu\not{\ell} - 2\left(-x\not{q}+z\not{p}\right)\gamma^\mu\left((1-x)\not{q}+z\not{p}\right)$$
$$+ \epsilon\left((1-x)\not{q}+z\not{p}\right)\gamma^\mu\left(-x\not{q}+z\not{p}\right)$$
$$+ 4m\left((1-x)q^\mu + zp^\mu\right) - \epsilon m\left((1-x)\not{q}+z\not{p}\right)\gamma^\mu$$
$$+ 4m\left(-xq^\mu + zp^\mu\right) - \epsilon m\,\gamma^\mu\left(-x\not{q}+z\not{p}\right)$$
$$-(2-\epsilon)\gamma^\mu m^2$$

$$= -(2-\epsilon)\,\not{\ell}\gamma^\mu\not{\ell} + 2x(1-x)\,\not{q}\gamma^\mu\not{q} - 2z(1-x)\,\not{p}\gamma^\mu\not{q}$$
$$+2xz\,\not{q}\gamma^\mu\not{p} - 2z^2\,\not{p}\gamma^\mu\not{p}$$
$$+\epsilon(1-x)(-x)\,\not{q}\gamma^\mu\not{q} - \epsilon z x\,\not{p}\gamma^\mu\not{q}$$
$$+\epsilon(1-x)z\,\not{q}\gamma^\mu\not{p} + \epsilon z^2\,\not{p}\gamma^\mu\not{p}$$
$$+4m\left((1-2x)q^\mu + 2zp^\mu\right)$$
$$-\epsilon m\left[\not{q}\gamma^\mu \cancel{} - x\not{q}\gamma^\mu - x\gamma^\mu\not{q} + z\not{p}\gamma^\mu + z\gamma^\mu\not{p}\right]$$
$$-(2-\epsilon)\gamma^\mu m^2 \quad ✓$$

Use $\not{\ell}\gamma^\mu\not{\ell}$
$$= \ell^\alpha\ell^\beta\gamma_\alpha\gamma^\mu\gamma_\beta$$
$$= \frac{1}{d}\ell^2 g^{\alpha\beta}\gamma_\alpha\gamma^\mu\gamma_\beta$$
$$= \frac{1}{d}\ell^2\gamma^\beta\gamma^\mu\gamma_\beta$$
$$= -\frac{1}{d}\ell^2(2-\epsilon)\gamma^\mu$$

$$= \frac{(2-\epsilon)^2}{d}\ell^2\gamma^\mu + (2-\epsilon)x(1-x)\,\not{q}\gamma^\mu\not{q}$$
$$-2z\,\not{p}\gamma^\mu\not{q} + (2-\epsilon)zx\,\not{p}\gamma^\mu\not{q}$$
$$+(2-\epsilon)xz\,\not{q}\gamma^\mu\not{p} - (2-\epsilon)z^2\,\not{p}\gamma^\mu\not{p}$$
$$+\epsilon z\,\not{q}\gamma^\mu\not{p}$$
$$+4m\left((1-2x)q^\mu + 2zp^\mu\right)$$
$$-\epsilon m\left(\not{q}\gamma^\mu - x\,2q^\mu + z\,2p^\mu\right)$$
$$-(2-\epsilon)\gamma^\mu m^2 \quad ✓$$

Drop pure $\not{q}$ terms.

$$= \frac{(2-\epsilon)^2}{d}\ell^2\gamma^\mu + (2-\epsilon)x(1-x)\gamma^\mu q^2$$
$$+2z\gamma^\mu\not{p}\not{q} - (2-\epsilon)zx\,\gamma^\mu\not{p}\not{q}$$
$$+(2-\epsilon)xz\,\not{q}\gamma^\mu m - (2-\epsilon)z^2\,\not{p}\gamma^\mu m$$
$$+\epsilon z\,\not{q}\gamma^\mu m$$
$$+4m\left((1-2x)q^\mu + 2zp^\mu\right)$$
$$-\epsilon m\left(\not{q}\gamma^\mu - 2xq^\mu + 2zp^\mu\right)$$
$$-(2-\epsilon)\gamma^\mu m^2 \quad ✓$$

$$
\begin{aligned}
= \frac{(2-\epsilon)^2}{d}\, \ell^2 \gamma^\mu \; & - (2-\epsilon)\, x(1-x)\, \gamma^\mu q^2 \\
& + 4z\, \gamma^\mu\, p\cdot q - 2z\, \gamma^\mu \not{q}\, m \\
& - (2-\epsilon)\, zx\, \gamma^\mu\, 2p\cdot q + (2-\epsilon)\, zx\, \gamma^\mu \not{q}\, m \\
& + (2-\epsilon)\, xz \not{q}\, \gamma^\mu m - (2-\epsilon)\, z^2\, 2p^\mu m + (2-\epsilon)\, z^2 \gamma^\mu m^2 \\
& + \epsilon z \not{q}\, \gamma^\mu m \\
& + 4m\left((1-2x)\, q^\mu + 2z p^\mu\right) \\
& - \epsilon m \not{q}\, \gamma^\mu - \epsilon m\left(-2x q^\mu + 2z p^\mu\right) \\
& - (2-\epsilon)\, \gamma^\mu m^2 \quad \checkmark
\end{aligned}
$$

Use $2p\cdot q = -q^2$

$$
\begin{aligned}
\bar{u}(p') \not{q} &= \bar{u}(p')(\not{p}' - \not{p}) = \bar{u}(p')(m - \not{p}) \\
\not{q}\, u(p) &= (\not{p}' - \not{p})\, u(p) = (\not{p}' - m)\, u(p)
\end{aligned}
$$

$$
\begin{aligned}
= \frac{(2-\epsilon)^2}{d}\, \ell^2 \gamma^\mu \; & - (2-\epsilon)\, x(1-x)\, \gamma^\mu q^2 - 2z\, \gamma^\mu q^2 + (2-\epsilon)\, zx\, \gamma^\mu q^2 \\
& - 2z\, \gamma^\mu \not{p}' m + 2z m^2 \gamma^\mu + \\
& + (2-\epsilon)\, xz m\, 2q^\mu \\
& - (2-\epsilon)\, z^2\, 2p^\mu m + (2-\epsilon)\, z^2 \gamma^\mu m^2 + \epsilon m^2 z\, \gamma^\mu - \epsilon m z \not{p}\, \gamma^\mu \\
& + 4m\left((1-2x)\, q^\mu + 2z p^\mu\right) \\
& - \epsilon m^2 \gamma^\mu + \epsilon m \not{p}\, \gamma^\mu + 2\epsilon m\left(x q^\mu - z p^\mu\right) - (2-\epsilon)\, \gamma^\mu m^2 \quad \checkmark
\end{aligned}
$$

$$
\begin{aligned}
= \frac{(2-\epsilon)^2}{d}\, \ell^2 \gamma^\mu \; & + \gamma^\mu q^2 \left(-(2-\epsilon)\, x(1-x) - 2z + (2-\epsilon)\, zx \right) \\
& - 4z\, p'^\mu m + 2z m^2 \gamma^\mu + 2z m^2 \gamma^\mu \\
& + (2-\epsilon)\, xz m\, 2q^\mu - (2-\epsilon)\, z^2\, 2p^\mu m + (2-\epsilon)\, z^2 \gamma^\mu m^2 \\
& + \epsilon z m^2 \gamma^\mu - 2\epsilon m z p^\mu + \epsilon m^2 z\, \gamma^\mu \\
& + 4m(1-2x)\, q^\mu + 8 m z p^\mu \\
& - \epsilon m^2 \gamma^\mu + 2\epsilon m p^\mu - \epsilon m^2 \gamma^\mu + 2\epsilon m x q^\mu - 2\epsilon m z p^\mu \\
& - (2-\epsilon)\, \gamma^\mu m^2 \quad \checkmark
\end{aligned}
$$

Collect like terms:

$$= \frac{(2-\epsilon)^2}{d} \ell^2 \gamma^\mu + \gamma^\mu q^2 \left((2-\epsilon) x (-1+x+z) - 2z \right)$$
$$+ m p'^\mu (-4z)$$
$$+ m^2 \gamma^\mu \left(4z + (2-\epsilon) z^2 + 2z - 2\epsilon - (2-\epsilon) \right)$$
$$+ m q^\mu \left(2xz(2-\epsilon) + 4(1-2x) + 2x\epsilon \right)$$
$$+ m p^\mu \left(-2(2-\epsilon) z^2 - 2\epsilon z + 8z + 2\epsilon - 2\epsilon z \right) \quad \checkmark$$

$$= \frac{(2-\epsilon)^2}{d} \ell^2 \gamma^\mu + \gamma^\mu q^2 \left((2-\epsilon) x (-y) - 2z \right)$$
$$+ m p'^\mu (-4z)$$
$$+ m^2 \gamma^\mu \left(-2 + 4z + 2z^2 + \epsilon(-z^2 + 2z - 1) \right)$$
$$+ m p^\mu \left(-4z^2 + 8z + \epsilon(2 - 4z + 2z^2) \right)$$
$$+ m q^\mu \left(4 - 8x + 4xz + \epsilon(2x - 2xz) \right) \quad \checkmark$$

Reshuffle to get $p^\mu + p'^\mu$ term & then use Gordon identity.

$$= (2-\epsilon)^2 \frac{\ell^2}{d} \gamma^\mu + \gamma^\mu q^2 \left(-(2-\epsilon) xy - 2z \right)$$
$$+ m (p+p')^\mu (-4z)$$
$$+ m^2 \gamma^\mu \left(-2 + 4z + 2z^2 + \epsilon(-z^2 + 2z - 1) \right)$$
$$+ m p^\mu \left(-4z^2 + 12z + \epsilon(2 - 4z + 2z^2) \right)$$
$$+ m q^\mu \left(4 - 8x + 4xz + \epsilon(2x - 2xz) \right)$$

Use $q^\mu = p'^\mu - p^\mu$, so $p^\mu = \frac{1}{2}(p'^\mu + p^\mu) - \frac{1}{2} q^\mu$

$$= (2-\epsilon)^2 \frac{\ell^2}{d} \gamma^\mu - \gamma^\mu q^2 \left((2-\epsilon) xy + 2z \right)$$
$$+ m (p+p')^\mu \left(-4z - 2z^2 + 6z + \epsilon(1 - 2z + z^2) \right)$$
$$+ m^2 \gamma^\mu \left(-2 + 4z + 2z^2 + \epsilon(-1 + 2z - z^2) \right)$$
$$+ m q^\mu \big(4 - 8x + 4xz + 2x\epsilon(1-z)$$
$$+ 2z^2 - 6z + \epsilon(-1 + 2z - z^2) \big)$$

$$= (2-\epsilon)^2 \frac{\ell^2 \gamma^\mu}{d} - \gamma^\mu q^2 \left((2-\epsilon) xy + 2z \right)$$
$$+ m (p+p')^\mu \left(2z - 2z^2 + \epsilon (1 - 2z + z^2) \right)$$
$$+ m^2 \gamma^\mu \left(-2 + 4z + 2z^2 + \epsilon (-1 + 2z - z^2) \right)$$
$$+ m q^\mu \left(4 - 8x - 6z + 4xz + 2z^2 + \epsilon (-1 + 2z - z^2 + 2x - 2xz) \right)$$

Can Check that q^μ term vanishes after integration over Feynman parameters, as needed for Ward identity.

Explicitly, $\Delta = -xy q^2 + (z-1)^2 m^2 + z\mu^2$ and is symmetric for $x \leftrightarrow y$.

Coeff. of $q^\mu = 2(y-x)(2-z) + \epsilon(1-z)(x-y)$ and so is antisymmetric for x, y integration & vanishes.

Now, use Gordon identity:

$$\bar{u}(p') \gamma^\mu u(p) = \bar{u}(p') \left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] u(p)$$

$$= (2-\epsilon)^2 \frac{\ell^2 \gamma^\mu}{d} - q^2 \gamma^\mu \left((2-\epsilon) xy + 2z \right)$$
$$+ m^2 \gamma^\mu \left(-2 + 4z + 2z^2 + \epsilon(-1 + 2z - z^2) + 4z - 4z^2 + \epsilon (2 - 4z + 2z^2) \right)$$
$$+ i\sigma^{\mu\nu} q_\nu m \left(2z(1-z) + \epsilon (1-z)^2 \right)$$

$$= (2-\epsilon)^2 \frac{\ell^2}{d} \gamma^\mu - q^2 \gamma^\mu \left(2(z + yx) - \epsilon xy \right)$$
$$+ m^2 \gamma^\mu \left(-2 + 8z - 2z^2 + \epsilon (1 - 2z + z^2) \right)$$
$$+ i\sigma^{\mu\nu} q_\nu m (1-z) \left(2z + \epsilon (1-z) \right)$$

Reuse ℓ & $\Delta \equiv -xyq^2 + (z-1)^2 m^2 + z\mu_\kappa^2$ explicit photon mass regulator

Thus,
$$iM = -e^3 \epsilon_\mu(q) \int dx\,dy\,dz\, \delta(x+y+z-1)$$
$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\bar{u}(p')\Big[(2-\epsilon)^2 \frac{\ell^2}{d}\gamma^\mu - q^2\gamma^\mu\big(2(z+yx)-\epsilon xy\big) + m^2\gamma^\mu\big(-2(1-4z+z^2)+\epsilon(1-z)^2\big) + i\sigma^{\mu\nu}q_\nu m(1-z)\big(2z+\epsilon(1-z)\big)\Big]u(p)}{(\ell^2-\Delta+i\epsilon)^3}$$

Use
$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2-\Delta)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-\frac{d}{2})}{2} \left(\frac{1}{\Delta}\right)^{2-d/2}$$
&
$$\int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2-\Delta)^3} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(3-d/2)}{2} \left(\frac{1}{\Delta}\right)^{3-d/2}$$

$$iM = -e^3 \epsilon_\mu(q) \cdot \int dx\,dy\,dz\, \delta(x+y+z-1)$$
$$\bar{u}(p')\Big[(2-\epsilon)^2 \frac{1}{d}\cdot\gamma^\mu\left(\frac{i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-d/2)}{2}\left(\frac{1}{\Delta}\right)^{2-d/2}\right)$$
$$+\Big(-q^2\gamma^\mu\big(2(z+yx)-\epsilon xy\big) + m^2\gamma^\mu\big(-2(1-4z+z^2)+\epsilon(1-z)^2\big) + i\sigma^{\mu\nu}q_\nu m(1-z)\big(2z+\epsilon(1-z)\big)\Big)$$
$$\frac{-i}{(4\pi)^{d/2}}\frac{\Gamma(3-d/2)}{2}\left(\frac{1}{\Delta}\right)^{3-d/2}\Big]u(p)$$

$$iM = \frac{-e^3}{2}\epsilon_\mu(q)\int dx\,dy\,dz\,\delta(x+y+z-1)\frac{i}{(4\pi)^{d/2}}$$
$$\bar{u}(p')\Big[(2-\epsilon)^2\frac{1}{2}\cdot\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\cdot\gamma^\mu$$
$$+\frac{\Gamma(3-d/2)}{\Delta^{3-d/2}}\cdot\Big(q^2\gamma^\mu\big(2(z+yx)-\epsilon xy\big) + m^2\gamma^\mu\big(+2(1-4z+z^2)-\epsilon(1-z)^2\big) - i\sigma^{\mu\nu}q_\nu m(1-z)\big(2z+\epsilon(1-z)\big)\Big]u(p)$$

This is the one-loop addition to the electron vertex with the photon in QED. (in dim. reg.)