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Lecture 22.
Quantum Field Theory I.
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June 28, 2021

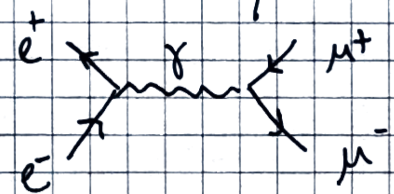
Last time: $e^+e^- \to \mu^+\mu^-$ tree-level
Today: $e^+e^- \to q\bar{q}$ / R ratio, crossing symmetry, Compton scattering.

Note: will write longer lecture notes with calculations but not detail all steps ~~do~~ during the lecture.

At the end of last time, we derived the tree-level calculation for $e^+e^- \to \mu^+\mu^-$ cross section:

$$\sigma_{tot}(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3E_{CM}^2}\sqrt{1-\frac{4m_\mu^2}{E_{CM}^2}}\left(1+\frac{2m_\mu^2}{E_{CM}^2}\right)$$

We had only one diagram:



Now, suppose we replace the muon by a different fermion. We would then have 3 changes:

① Electric charge: $(-ie\gamma^\nu)$ Feynman rule becomes $(ieQ_f\gamma^\nu)$, so cross section gets multiplied by Q_f^2.

② Mass: $m_\mu \to m_f$ everywhere.

③ Sum over final ~~f~~ states can include a new factor.

In the case of quarks, we would have a "color factor," since a given quark is not just labeled by spin but also by color.

Warning: These rescalings + replacements work as long as there are <u>not</u> new diagrams/interactions; replacing μ by e is Bhabha scattering.

So, for a given quark, we have

$$\sigma_{tot}(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3E_{CM}^2}\cdot Q^2 \cdot N_c \sqrt{1-\frac{4m_q^2}{E_{CM}^2}}\left(1+\frac{2m_q^2}{E_{CM}^2}\right)$$

Note that for E_{CM} large, we can ignore the phase space factor (the square root) and the finite $\frac{2m_q^2}{E_{CM}^2}$ contribution.

Then, $\sigma_{tot}(\mu^+\mu^-) \approx \frac{4\pi\alpha^2}{3E_{CM}^2}$ and

$$\sigma_{tot}(q\bar{q}) \approx \frac{4\pi\alpha^2}{3E_{CM}^2} Q^2 N_c = Q^2 N_c \cdot \sigma(\mu^+\mu^-)$$

So, for a large enough CoM collision energy, the total $q\bar{q}$ cross section is a sum over accessible quark pairs, weighted by the electric charge squared and the number of colors, and it can be normalized to the $\sigma_{tot}(\mu^+\mu^-)$ rate.

This is directly related to a much more complicated cross section, $\sigma(e^+e^- \to \text{hadrons})$.

Phenomenologically, we have not observed free colored quarks, because the strong interaction binds colored objects into colorless bound states at very short distances and very fast timescales (for temperature $= 0$).

The colorless bound states we have observed are called hadrons, generally, and there are two categories:

(A) mesons, which are bound states of a quark + anti-quark (opposite color charge, not necessarily flavor, or EM charge)

(B) baryons, which are antisymmetric combinations of three quarks or three antiquarks (again, "anti" only refers to color).

Thus, when we measure $\sigma(e^+e^- \to \text{hadrons})$, we can essentially identify the cross section with $\sigma(e^+e^- \to q\bar{q})$, where we *inclusively* sum up all hadronic particle production that we see in our detectors. Although the cross section will peak (with narrow spikes in rate) when the E_{cm} matches a resonant $q\bar{q}$ particle (like the J/ψ or $\psi(2S)$ or $\Upsilon(1S, 2S, 3S)$ states; which is when the photon propagator matches a resonant mass in the final state), the continuum differential cross section as a function of E_{cm} will show discrete jumps whenever we reach a new pair production threshold of a new quark. By comparing to the $\mu^+\mu^-$ cross section, we can then determine the $Q^2 N_c$ of the new quark. See Figure 5.3 of P+S.

We can also study the helicity structure of $e^+e^- \to \mu^+\mu^-$ scattering, by *not* summing over final state spins but instead considering fixed polarizations of incoming + outgoing electrons + muons. The calculation is readily done by recognizing that $\gamma^\mu = \gamma^\mu(P_L + P_R)$, and we can treat $\gamma^\mu P_L$ or $\gamma^\mu P_R$ separately in the matrix element and also use the projection operators as subscripts on the final / initial state fermions.

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For $e^-_R e^+_L \to \mu^-_R \mu^+_L$, we need P_R projections on the fermion lines (incoming e^- & outgoing μ^-), which changes the matrix element squared to

$$|\mathcal{M}|^2 = e^4 (1+\cos\theta)^2$$

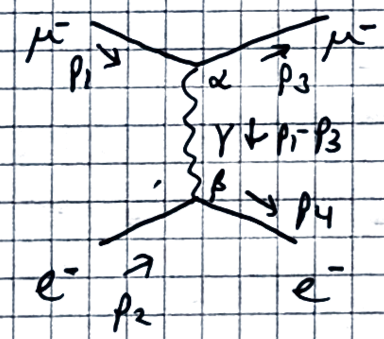
$$\Rightarrow \frac{d\sigma}{d\Omega}(e^-_R e^+_L \to \mu^-_R \mu^+_L) = \frac{\alpha^2}{4E_{CM}}(1+\cos\theta)^2$$

$$\frac{d\sigma}{d\Omega}(e^-_R e^+_L \to \mu^-_L \mu^+_R) = \frac{\alpha^2}{4E_{CM}}(1-\cos\theta)^2$$

Can check QED is parity invariant, so $L \leftrightarrow R$ in each expression is the same $\frac{d\sigma}{d\Omega}$. All others are zero, and the total sum (with appropriate averaging of initial states) reproduces the σ_{tot} we had before (with $m_\mu \to 0$).

Crossing Symmetry

Suppose we wanted to calculate a related process, $e^-\mu^- \to e^-\mu^-$ scattering.



$$i\mathcal{M} = \bar{u}(p_3)\cdot(-ie\gamma^\alpha)u(p_1)\cdot\left(\frac{-ig_{\alpha\beta}}{(p_1-p_3)^2+i\epsilon}\right)\cdot\bar{u}(p_4)(-ie\gamma^\beta)u(p_2)$$

$$= \frac{ie^2}{(p_1-p_3)^2+i\epsilon}\,\bar{u}(p_3)\gamma^\alpha u(p_1)\,\bar{u}(p_4)\gamma_\alpha u(p_2)$$

We can square & calculate the traces & avg. over initial spins

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4(p_1-p_3)^4}\, Tr[\gamma^\alpha(\not p_1+m_\mu)\gamma^\mu(\not p_3+m_\mu)]\; Tr[\gamma_\alpha(\not p_2+m_e)\gamma_\mu(\not p_4+m_e)]$$

Compare to $e^+e^-\to\mu^+\mu^-$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4(p_1+p_2)^4}\, Tr[\gamma_\mu(\not p_4-m_\mu)\gamma_\alpha(\not p_3+m_\mu)]\; Tr[\gamma^\mu(\not p_2+m_e)\gamma^\alpha(\not p_1-m_e)]$$

Note that we can get the $e^-\mu^-\to e^-\mu^-$ amplitude by the replacement rule: (working from $e^+e^-\to\mu^+\mu^-$, lower expression):

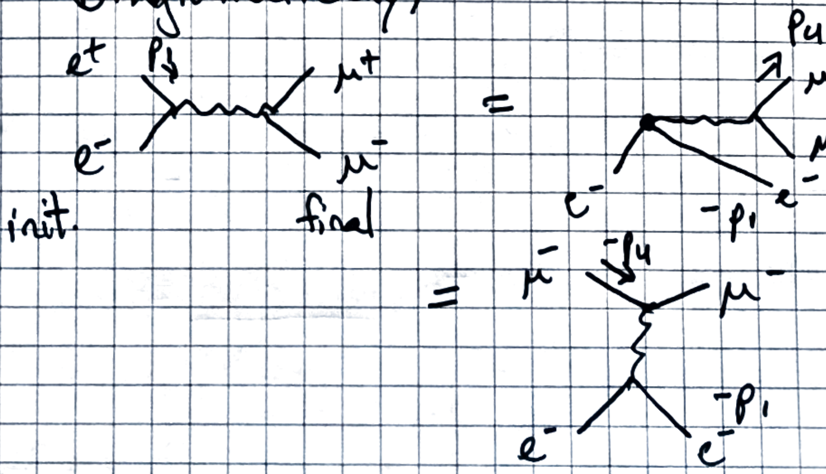
$$p_1\to -p_4,\quad p_2\to p_2,\quad p_3\to p_3,\quad p_4\to -p_1$$

Check denominator:

$$(p_1+p_2)^2\to(-p_4+p_2)^2=(p_1-p_3)^2 \quad \text{by} \quad p_1+p_2=p_3+p_4 \;\Leftrightarrow\; p_2-p_4=p_3-p_1$$

This is an example of crossing symmetry, which is the statement that the amplitude for an incoming particle with momentum p is the same as the same process where the particle is replaced by an antiparticle in the final state with momentum $-p$.

Diagramatically,

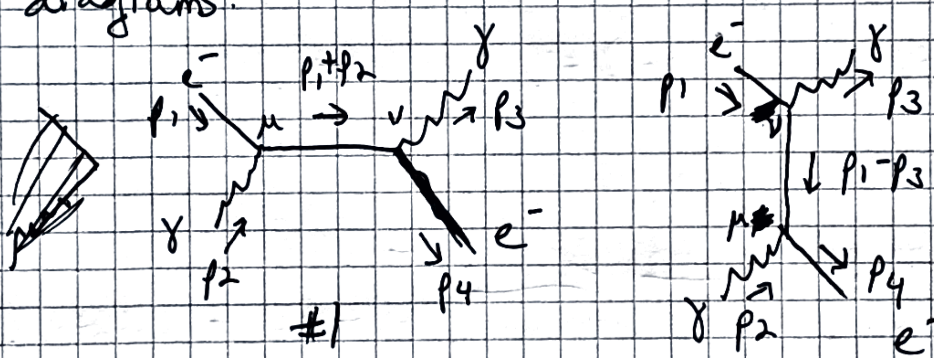


to get from one process to the other.

Crossing symmetry is very useful because fundamentally, fields at vertices can act as creation of outgoing particles = annihilation of incoming antiparticles while conjugate fields act as creation of outgoing antiparticles = annihilation of incoming particles. So crossing symmetry is built into the structure of QFT amplitudes.

Compton scattering

The next tree-level QED scattering we consider is Compton scattering, $e^-\gamma \to e^-\gamma$, which has two diagrams:



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The fermion parts of the two diagrams are identical, only the bosonic photon lines need to change in order to get from one diagram to the next, so no relative sign.

We write the matrix elements:

$$i\mathcal{M}_{tot} = i\mathcal{M}_1 + i\mathcal{M}_2$$

$$i\mathcal{M}_1 = \epsilon^*_\nu(p_3)\cdot\epsilon_\mu(p_2)\cdot\bar{u}(p_4)\cdot(-ie\gamma^\nu)\cdot\frac{i(\not{p}_1+\not{p}_2+m_e)}{(p_1+p_2)^2-m_e^2+i\epsilon}\cdot(-ie\gamma^\mu)\,u(p_1)$$

$$i\mathcal{M}_2 = \epsilon^*_\nu(p_3)\, \epsilon_\mu(p_2)\, \bar{u}(p_4)\, (-ie\gamma^\mu)\, \frac{i(\not{p}_1 - \not{p}_3 + m_e)}{(p_1-p_3)^2 - m_e^2 + i\epsilon}$$
$$(-ie\gamma^\nu)\, u(p_1)$$

<u>Note:</u> All external states should be labeled with same Lorentz indices & momenta to make it clear that the <u>amplitudes</u> <u>interfere</u>.

We have again left spin indices on the spinors implicit.

We will discard the $+i\epsilon$ in the fermion propagators since this is a tree-level process.

Reorganize:

$$i\mathcal{M}_1 = \epsilon^*_\nu \epsilon_\mu \frac{(-ie^2)}{(p_1+p_2)^2 - m_e^2} \quad \bar{u}_4 \gamma^\nu (\not{p}_1 + \not{p}_2 + m_e)\gamma^\mu u_1$$

$$i\mathcal{M}_2 = \epsilon^*_\nu \epsilon_\mu \frac{(-ie^2)}{(p_1-p_3)^2 - m_e^2} \quad \bar{u}_4 \gamma^\mu (\not{p}_1 - \not{p}_3 + m_e)\gamma^\nu u_1$$

Use $(p_1+p_2)^2 - m_e^2 = m_e^2 + 2p_1\cdot p_2 - m_e^2 = 2p_1\cdot p_2$

$$(p_1-p_3)^2 - m_e^2 = m_e^2 - 2p_1\cdot p_3 - m_e^2 = -2p_1\cdot p_3$$

Also, we can use the Dirac eqn:

$$(\not{p}_1 + \not{p}_2 + m_e)\gamma^\mu u_1 = (2p_1^\mu - \gamma^\mu \not{p}_1 + \not{p}_2\gamma^\mu + m_e\gamma^\mu)u_1$$

$\not{p}_1\gamma^\mu + \gamma^\mu\not{p}_1 = 2p_1^\mu$ but $(\not{p}_1 - m_e)u_1 = 0$

$$\Rightarrow \quad = (2p_1^\mu + \not{p}_2\gamma^\mu)u_1$$

$$(\not{p}_1 - \not{p}_3 + m_e)\gamma^\nu u_1 = (2p_1^\nu - \gamma^\nu\not{p}_1 - \not{p}_3\gamma^\nu + m_e\gamma^\nu)u_1$$
$$= (2p_1^\nu - \not{p}_3\gamma^\nu)u_1$$

We get

$$i\mathcal{M}_{tot} = -ie^2 \epsilon_\nu^* \epsilon_\mu \left[\bar{u}_4 \frac{\gamma^\nu (2p_1^\mu + \not{p}_2 \gamma^\mu) u_1}{2p_1 \cdot p_2} + \bar{u}_4 \frac{\gamma^\mu (2p_1^\nu - \not{p}_3 \gamma^\nu) u_1}{-2p_1 \cdot p_3} \right]$$

$$= -ie^2 \epsilon_\nu^* \epsilon_\mu \left[\bar{u}_4 \left(\frac{\gamma^\nu \not{p}_2 \gamma^\mu + 2p_1^\mu \gamma^\nu}{2p_1 \cdot p_2} + \frac{-\gamma^\mu \not{p}_3 \gamma^\nu + 2p_1^\nu \gamma^\mu}{-2p_1 \cdot p_3} \right) u_1 \right]$$

Will square + finish cross section next time.