Lecture 19.
Quantum Field Theory I.
Felix Yu
June 16, 2021.

Last time: LSZ reduction formula
Today: Feynman rules for fermions, Yukawa theory, begin QED.

We now have a framework for calculating cross sections and decay widths, where the remaining job is ~~the~~ the calculation of the corresponding matrix elements that correspond to the desired amplitude and process.

In general, we can categorize processes according to the number of external particles, vertices, and loops. At the moment, we will start with processes that occur without any loops, i.e. without the contribution of virtual particles with undetermined momenta that needs to be integrated over.

We will also (almost) exclusively work with momentum space Feynman rules, where final state scalars have a factor of 1 instead of $e^{-ip\cdot x}$ and vertices are just the coupling strength (like $-i\lambda$) instead of the coupling strength multiplied by $\int d^4x$, and propagators are the Fourier transform of $D_F(x-y)$ instead of simply $D_F(x-y)$. When in doubt, you can usually look up Feynman rules or perform the FT from position space.

We will also conventionally draw diagrams with time flowing from left to right, so incoming particles / initial states will be on the left of the graph and outgoing / final states are on the right side. Note this is different from the convention in P+S, where diagrams generally have time flowing upward.

To get started with tree-level processes, we need to establish Feynman rules for fermions.

The basic Feynman rules for external (either incoming or outgoing) fermions and propagating fermions are essentially universal (just note that we always work with 4-component notation).

To find the propagator, recall

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)} = \langle 0 | T\psi(x)\bar{\psi}(y)|0\rangle$$

where time-ordering of fermions included an extra minus sign when the fermions were reversed:

$$T(\psi(x)\bar{\psi}(y)) = \begin{cases} \psi(x)\bar{\psi}(y) & \text{for } x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) & \text{for } x^0 < y^0. \end{cases}$$

More generally, in an n-pt. correlation fcn, the time-ordered product gets an overall factor of (-1) if the number of rearrangements is odd + (+1) if the number of rearrangements is even (dictated by the time ordering).

Recall that we then reexpressed the time ordered product as the sum of the normal ordered product plus the set of all possible contractions (see Wick's theorem). Then, since the normal ordering also counts $(-1)$ factors for each interchange of ladder operators, and since normal ordering pushes all creation operators to the left of annihilation operators, we have an unchanged Wick's theorem that applies to fermions!

$$T\left[\psi_1\bar{\psi}_2\psi_3\ldots\right] = N\left[\psi_1\bar{\psi}_2\psi_3\ldots + \text{all possible contractions}\right]$$

Note any odd # of fermion fields has vanishing correlation fcns.

Also, pairwise contractions are either 0 or $S_F(x-y)$:

$$\overbrace{\psi(x)\,\bar{\psi}(y)} = S_F(x-y)$$

$$\overbrace{\psi(x)\,\psi(y)} = 0$$

$$\overbrace{\bar{\psi}(x)\,\bar{\psi}(y)} = 0.$$

We also need the contraction of interacting fields on asymptotic states, which are denoted by momentum + spin:

$$\overbrace{\psi_I(x)\,|\vec{p},s\rangle} = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_{s'} a_{p'}^{s'} u^{s'}(p')e^{-ip'\cdot x}\sqrt{2E_p}\,a_p^{s\dagger}|0\rangle$$

$$= e^{-ip\cdot x}\,u^s(p)\,|0\rangle$$

from $\left\{a_p^r, a_q^{s\dagger}\right\} = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\,\delta^{rs}$

We can also use $\psi_I(x)$ to contract on the left to give $v^s(k)$, while $\bar{\psi}_I(x)$ acting to the right + left gives $\bar{v}^s(k)$ and $\bar{u}^s(k)$, respectively.

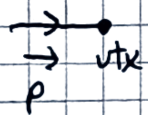
We summarize the standard propagator & external leg Feynman rules for fermions:

In momentum space,

propagator

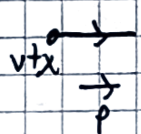

$$\frac{i(\not p + m)}{p^2 - m^2 + i\epsilon}$$

incoming ext. fermion



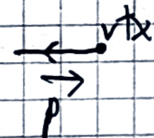

$u^s(p)$

outgoing ext. fermion



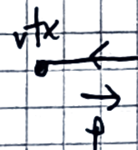

$\bar{u}^s(p)$

incoming ext. antifermion




$\bar{v}^s(p)$

outgoing ext. antifermion




$v^s(p)$

Convention: fermion lines solid, scalar lines dashed.

Tips + tricks:

1. Arrows on fermion lines show the direction of fermion flow, where fermions have fermion particle # = +1 and antifermions have particle # = -1. So an incoming fermion has an arrow pointing towards the vertex and an incoming anti-fermion has the arrow pointing away from the vertex. Likewise, the outgoing fermion has an arrow pointing away from the vtx & outgoing antifermion has arrow pointing into vertex.

2. Incoming & outgoing generally refer to the momentum direction, not necessarily the fermion flow.

3 By Lorentz symmetry, fermion flow must be continuous through a diagram, so you should always have arrows flowing from one external state through the diagram to terminate on another external state, or you have a closed loop of fermions.

Corollary A: You cannot have a vertex where an odd # of fermion lines meet.
All vertices have an even # of fermion lines.

B Aside: Majorana fermions can reverse fermion flow, allowing $\rightarrow\otimes\leftarrow$ "clashing" arrows, but we will not ~~consider~~ focus on calculations with Majorana fermions.

4 Closed loop of fermions gets a $(-1)$ factor.

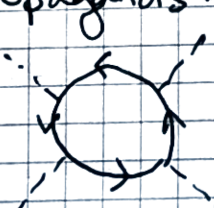
5 Swapping two external legs of identical fermions gives a relative factor of $-1$.

These last two rules result from the convention that $|\vec{p},\vec{k}\rangle \sim a_p^\dagger a_k^\dagger |0\rangle$, $\langle \vec{p}',\vec{k}'| \sim \langle 0| a_{k'} a_{p'}$

so that $(|p,k\rangle)^\dagger = \langle p,k|$.

(Otherwise there is an ambiguity about $(-1)$ factors).

Also, related to #4, a closed loop of fermions gives a trace over the numerator of the fermion propagators:



$$= \bar\psi\,\psi\bar\psi\,\psi\bar\psi\,\psi\bar\psi\,\psi$$

$$= (-1)\,\mathrm{tr}\left[\psi\bar\psi\,\psi\bar\psi\,\psi\bar\psi\,\psi\bar\psi\right]$$

$$= (-1)\,\mathrm{tr}\left[S_F\, S_F\, S_F\, S_F\right] \quad \text{(in pos. space)}$$

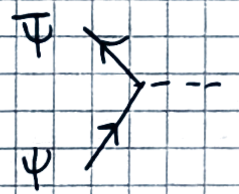
## Ex. Yukawa theory.

We have now set up the Feynman rules for fermions, but now we need an interaction. We start with Yukawa theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m_s^2}{2}\phi^2 + \bar{\Psi} i \cancel{\partial} \Psi - m_f \bar{\Psi}\Psi - y\phi\bar{\Psi}\Psi$$

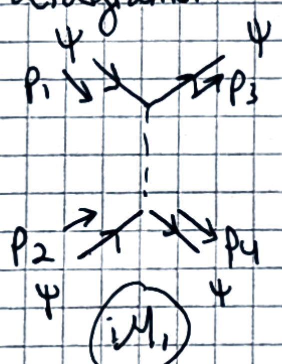
where $y$ is real. Yukawa interaction = scalar coupling to (scalar) fermion bilinear.

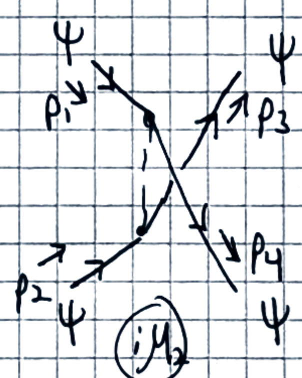
The basic Feynman rule for the interaction is




$-iy$

Practice: Consider 2-to-2 scattering of fermions. We have two diagrams.




The total matrix element is

$$iM_{tot} = iM_1 \underset{\text{because of antisymmetrization}}{-} iM_2$$

$$iM_1 = \bar{u}^s(p_3)(-iy)\cdot u^s(p_1)\cdot \bar{u}^{s'}(p_4)(-iy)u^{s'}(p_2) \cdot \frac{i}{(p_1-p_3)^2 - m_s^2 + i\epsilon}$$

$$iM_2 = \bar{u}^s(p_3)(-iy)\, u^s(p_2)\, \bar{u}^{s'}(p_4)(-iy)\, u^{s'}(p_1)\, \frac{i}{(p_1-p_4)^2 - m_s^2 + i\epsilon}$$

Tips + tricks

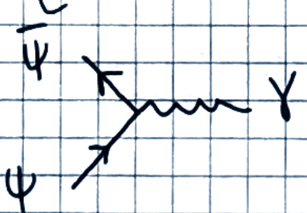
6. Note that we start on an ext. fermion line + work backwards against fermion number flow. This guarantees that we end up with a complete contraction over spin space: $\bar{u}^s(p) \ldots u^s(p)$ or equivalently, $\bar{v}^s(p) \ldots u^s(p)$ + other iterations.

Corollary! Since $\bar{u}^s(p_3)\, u^s(p_1)$ is fully contracted, we can <u>commute</u> it around the other factors. This also means we could have started with either fermion line: the order of $\bar{u}(p_3)\, u(p_1)$ vs. $\bar{u}(p_4)\, u(p_2)$ does not matter.

The Yukawa coupling is an identity in spin space and can be factored out.

But, we could instead have had a nontrivial fermion bilinear like a $\gamma^\mu$ matrix ~~~~ on the vertex, in which case we ~~~~ must leave the $\gamma$-matrix inside the spinor space contraction.

In fact, this is the coupling we encounter in quantum electrodynamics, QED, which will be the focus of the next few lectures.




$-ie\gamma^\mu$