Lecture 18.
Quantum Field Theory 1.
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June 14, 2021.

Last time: Kinematics of collisions + phase space integrals, LSZ reduction

Today: LSZ reduction formula.

In the last lecture, we concluded that (in position space)

$$i\mathcal{M}(p_1, \dots p_m \to p_1' \dots p_n')\,(2\pi)^4\,\delta^{(4)}(p_1 + \dots + p_m - p_1' \dots p_n')$$
$$= \frac{(\sum \text{ fully connected, amputated Feynman diagrams with } p_1, \dots p_m \text{ incoming} + p_1' \dots p_n' \text{ outgoing})}{Z_1^{1/2} \dots Z_m^{1/2} \quad Z_1^{1/2} \dots Z_n^{1/2}}$$

We will justify this claim in this lecture.

Follow Schwartz, p. 70-74.

Start with intial states at $t \to -\infty$ and final states at $t \to +\infty$, as asymptotic states participating in the scattering. We have

$$\underset{t=+\infty}{\langle f|}\, S - \mathbb{1}\, \underset{t=-\infty}{|i\rangle} = i(2\pi)^4\,\delta^{(4)}(\Sigma_p)\,\mathcal{M}$$

We have asymptotic free states

$$|i\rangle = \sqrt{2E_1}\sqrt{2E_2}\;a_{p_1}^\dagger(-\infty)\,a_{p_2}^\dagger(-\infty)\,|\Omega\rangle$$
$$|f\rangle = \sqrt{2E_3}\sqrt{2E_4}\dots\sqrt{2E_n}\;a_{p_3}^\dagger(+\infty)\dots a_{p_n}^\dagger(+\infty)\,|\Omega\rangle$$

Assume $|f\rangle \neq |i\rangle$, so discard $\mathbb{1}$.

$$\langle f|S|i\rangle = 2^{n/2}\sqrt{E_1 \dots E_n}\;\langle\Omega|\,a_{p_3}(\infty)\dots a_{p_n}(\infty)\,a_{p_1}^\dagger(-\infty)\,a_{p_2}^\dagger(-\infty)\,|\Omega\rangle$$

To progress, need to prove

$$i\int d^4x\ e^{ip\cdot x}\ (\Box + m^2)\,\phi(x) = \sqrt{2E_p}\,\big(a_p(\infty) - a_p(-\infty)\big)$$

for asymptotic free states. Assume fields die off at $|\vec{x}| = \pm\infty$, so can integrate by parts in $\vec{x}$.

Step by step:

$$i\int d^4x\ e^{ip\cdot x}\ (\partial_t^2 - \vec{\partial}_x^{\,2} + m^2)\,\phi(x)$$

$$= i\int d^4x\ e^{ip\cdot x}\ (\partial_t^2 + \vec{p}^{\,2} + m^2)\,\phi(x)$$ IBP, require $(\partial_x \phi\, e^{-ip\cdot x})\Big|_{-\infty}^{\infty} = 0$

$$= i\int d^4x\ e^{ip\cdot x}\ (\partial_t^2 + E_p^2)\,\phi(x)$$

Now, use $\partial_t\left[e^{ip\cdot x}(i\partial_t + E_p)\,\phi(x)\right] = \left[iE_p e^{ip\cdot x}(i\partial_t + E_p) + e^{ip\cdot x}(i\partial_t^2 + E_p\partial_t)\right]\phi$

$$= i\,e^{ip\cdot x}(\partial_t^2 + E_p^2)\,\phi(x)$$ independent of BCs.

So, we get RHS above

$$= \int dt\ \partial_t\left[e^{iE_p t}\int d^3x\ e^{-i\vec{p}\cdot\vec{x}}\ (i\partial_t + E_p)\,\phi(x)\right]$$

We have total derivative on RHS, evaluate at $t = \pm\infty$

Note that $a_p(t)$ + $a_p^\dagger(t)$ operators are time-~~[illegible]~~ independent at ~~[illegible]~~ $t \mapsto \pm\infty$.

Calculate:

$$\int d^3x\ e^{-i\vec{p}\cdot\vec{x}}\ (i\partial_t + E_p)\,\phi(x)$$

Heisenberg pict. Put time-dependence on a + $a^\dagger$

$$= \int d^3x\ e^{-i\vec{p}\cdot\vec{x}}\ (i\partial_t + E_p)\int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left(a_k(t)\,e^{-ik\cdot x} + a_k^\dagger(t)\,e^{ik\cdot x}\right)$$

$$\star = \int \frac{d^3k}{(2\pi)^3}\int d^3x\left[\frac{(E_k + E_p)}{\sqrt{2E_k}}\,a_k(t)\,e^{-ik\cdot x}\,e^{-i\vec{p}\cdot\vec{x}} + \frac{(-E_k + E_p)}{\sqrt{2E_k}}\,a_k^\dagger(t)\,e^{ik\cdot x}\,e^{-i\vec{p}\cdot\vec{x}}\right]$$

using $\partial_t a_k(t) = 0$ for bdrys.

We integrate over d^3x: $\int d^3x\ e^{-ik\cdot x}\,e^{-i\vec{p}\cdot\vec{x}} = e^{-iE_k t}\,\delta^{(3)}(\vec{p} - \vec{k})\,(2\pi)^3$

$+ \int d^3x\ e^{+ik\cdot x}\,e^{-i\vec{p}\cdot\vec{x}} = e^{+iE_k t}\,\delta^{(3)}(\vec{p} + \vec{k})\,(2\pi)^3$

Then $\star = \sqrt{2E_p}\ a_p(t)\ e^{-iE_p t}$

$$\text{So, } i\int d^4x\, e^{ip\cdot x}(\Box + m^2)\phi(x) = \int dt\, \partial_t [\ldots]$$
$$= \sqrt{2E_p}\left(a_p(\infty) - a_p(-\infty)\right)$$

and by h.c.,

$$-i\int d^4x\, e^{-ip\cdot x}(\Box + m^2)\underset{\text{real field}}{\phi(x)} = \sqrt{2E_p}\left(a_p^\dagger(\infty) - a_p^\dagger(-\infty)\right)$$

Now insert into

$$\langle f|S|i\rangle = 2^{n/2}\sqrt{E_1 \cdots E_n}\,\langle\Omega|a_{p_3}(\infty)\ldots a_{p_n}(\infty)\,a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)|\Omega\rangle$$
$$= \sqrt{2^n E_1 \cdots E_n}\,\langle\Omega|T\{a_{p_3}\cdots a_{p_n}\,a_{p_1}^\dagger a_{p_2}^\dagger\}|\Omega\rangle$$
$$= \sqrt{2^n E_1 \cdots E_n}\,\langle\Omega|T\{[a_{p_3}(\infty) - a_{p_3}(-\infty)]\ldots[a_{p_n}(\infty) - a_{p_n}(-\infty)]$$
$$[a_{p_1}^\dagger(-\infty) - a_{p_1}^\dagger(+\infty)][a_{p_2}^\dagger(-\infty) - a_{p_2}^\dagger(\infty)]\}|\Omega\rangle$$

All new a_i will shuffle to annihilate on vacuum by time-ordering.

Replace ladder operators by differentials:

$$\langle f|S|i\rangle = \langle p_3 \ldots p_n|S|p_1 p_2\rangle$$
$$= \left[i\int d^4x_1\, e^{-ip_1\cdot x_1}(\Box_1 + m^2)\right]\cdots\left[i\int d^4x_n\, e^{ip_n\cdot x_n}(\Box_n + m^2)\right]$$
$$\langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle$$

This is the Lehmann-Symanzik-Zimmermann reduction formula.

Given LHS is not identity, we have

$$\langle f|S|i\rangle = i(2\pi)^4\delta^{(4)}(\Sigma p)\,\mathcal{M}(p_1 p_2 \to p_3 \ldots p_n)$$

The RHS, with the n-pt. correlation fcn., we have already becomes the sum of fully contracted diagrams.

Discussion:

- $[\int d^4x_1\, e^{-ip_1\cdot x_1} \ldots]$ is the integral version of selecting the ~~p~~ momentum particle state at position x_1 given that the $[\ldots]$ is an set of exponentials.

- $(\Box_1 + m^2)$ is a differential operator acting on $\langle\Omega| T\{\phi \ldots \phi\}|\Omega\rangle$. If we consider $\langle\Omega|T\{\ldots\}|\Omega\rangle$ as a function in complex momenta, then $(\Box_1+m^2)$ extracts the pole of the amplitude corresponding to the coeff. function of $\frac{1}{p_i^2 - m^2}$.

  There are two aspects to this pole extraction:

  (a) The <u>residue</u> of the amplitude, which is called the 'wavefcn. normalization (when we consider a free theory with interactions.) This is a Z_i factor:

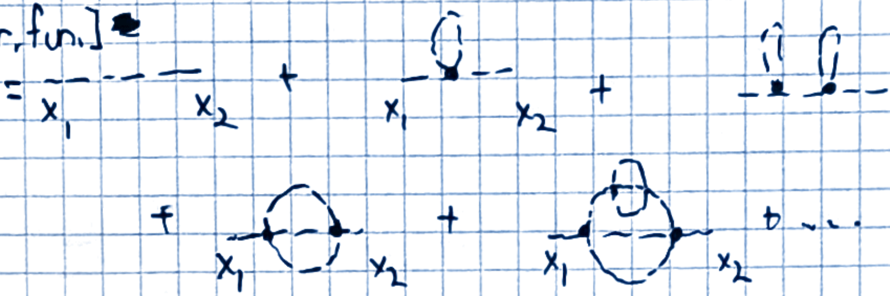
  $$\langle\Omega| T\{\phi(x_1)\,\phi(x_2)\}|\Omega\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{iZ}{k^2-m^2+i\epsilon} e^{ik\cdot(x_1-x_2)} + \ldots$$

  possible bd. states + multiparticle continuum.

  (b.) The <u>location</u> of the pole, which is called mass renormalization.

Diagramatically, these changes in the wavefcn + mass are given by the full correction to the 2-pt. correlation function by interactions.

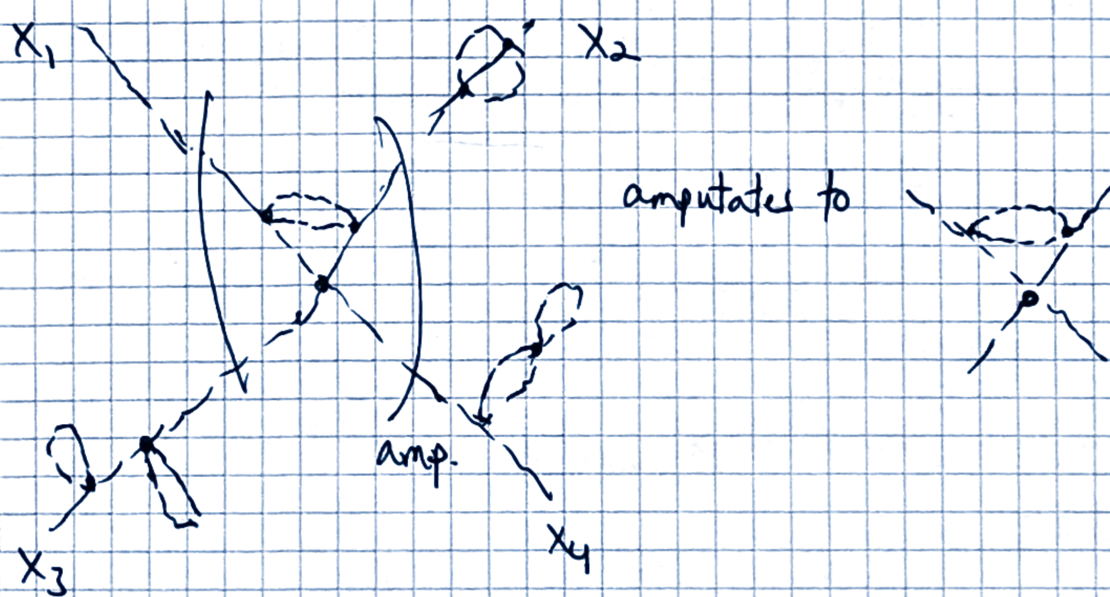
[In $\lambda\phi^4$ theory, for example, we have
[2-pt. corr. fcn.]



Since all of the diagrams apply only to external legs of our original ~~cor~~ n-pt. corr. fcn., we can <u>amputate</u> the n-pt. corr. fcn. diagrams and put all external leg corrections as simple Z_i and m corrections.

- For Z given by $\langle\Omega| T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle$, with two powers of ϕ, it is then appropriate to take the square root for each ϕ_i in the n-pt. corr. fcn.

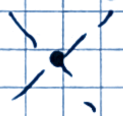
Ex. 4-pt. fcn.



Thus, in practice, we evaluate matrix elements by summing all connected, amputated diagrams.

In momentum space, $\frac{\lambda}{4!}\phi^4$ theory has the tree-level Feynman rules.

rules:  propagator $= \frac{i}{p^2-m^2+i\epsilon}$

 $= -i\lambda$

ext.  $= 1$

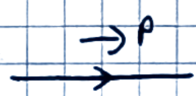
Momentum conserved at each vtx.

Integrate over undetermined loop momenta. $\int \frac{d^4k}{(2\pi)^4}$

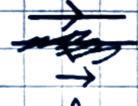
Divide by symmetry factor.

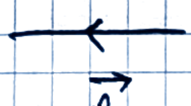
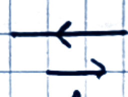
For fermions, Feynman rules are analogous, ~~exp~~ except for complications from spinor space and ~~s~~ antisymmetrization factors.

Fermion Feynman propagator:

1.  $\frac{i(\not p+m)}{p^2-m^2+i\epsilon}$

Arrows on lines indicate direction of fermion number flow.

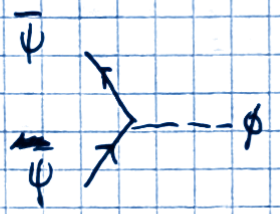
2. external fermion  incoming $u^s(p)$ |  outgoing $\bar{u}^s(p)$

3. antifermion  incoming $\bar{v}^s(p)$ |  outgoing $v^s(p)$

Simplest Fermion + scalar interaction:

Yukawa Lagrangian.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 + \bar{\psi}(i\not{\partial})\psi - y\phi\bar{\psi}\psi$$



$= -iy$

#4-3 Still have momentum cons. + integration over undetermined loop momenta.

Special rules for fermions:

1. Loop of fermions gives (-1) factor.
2. Switching two identical fermion lines (or identical antifermion lines) gives (-1) relative factor.