Lecture 17.
Quantum Field Theory 1.
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Last time: Cross sections + S-matrix, matrix elements, decay width
Today:

Last time, we had the expression for the differential cross section for $A+B \to n$ particles,

$$d\sigma = \left(\prod_{i=1}^{n} \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E_{\vec{p}_i}} \right) (2\pi)^4 \delta^{(4)}\left(p_A + p_B - \sum_{i=1}^{n} p_i\right)$$
$$\frac{1}{2E_A 2E_B} \frac{1}{|v_A - v_B|} \quad |\mathcal{M}(p_A, p_B \to p_1, p_2, \dots p_n)|^2$$

[P+S, 4.79]

Note the phase space integral + the matrix element $\mathcal{M}$ are separately Lorentz invariant, so the Lorentz transformation only comes from $\dfrac{1}{E_A E_B |v_A - v_B|} = \dfrac{1}{|p_A^z E_B - p_B^z E_A|}$ which is invariant under boosts in the z-direction.

Frequently, we will only have a 2-body phase space. In this case, it is useful to simplify the phase space integration.
We label $\int d\Pi_n = \left(\prod_{f=1}^{n} \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) (2\pi)^4 \delta^{(4)}\left(\Sigma p_{in} - \Sigma p_{final}\right)$

as the n-body phase space integral.

Thus, $$\int d\Pi_2 = \int \frac{d^3 p_1}{(2\pi)^3} \frac{1}{2E_1} \int \frac{d^3 p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^{(4)}(P_{tot} - p_1 - p_2)$$

where p_1 + p_2 are the 4-momenta of the final state particles.

This can be evaluated in any frame, and it is simplest to do so in the center-of-mass frame, so $\vec{p}_1$ and $\vec{p}_2$ are back-to-back. The $\delta^{(3)}(\vec{P}-\vec{p}_1-\vec{p}_2)$ integration has ~~gives~~ $\vec{P}=0$ in this frame & so $\vec{p}_1 = -\vec{p}_2$, which eliminate the $\int d^3p_2$ integrals.

$$\Rightarrow \int d\Pi_2 = \int \frac{dp_1\, p_1^2\, d\Omega}{(2\pi)^3\, 2E_1\, 2E_2}(2\pi)\,\delta(E_{cm}-E_1-E_2)$$

where $p_1 = |\vec{p}_1|$, $E_1 = \sqrt{m_1^2+|\vec{p}_1|^2}$, $E_2 = \sqrt{m_2^2+|\vec{p}_1|^2}$ since $|\vec{p}_1| = |\vec{p}_2|$

We have $\displaystyle\int \frac{dp_1\, p_1^2\, d\Omega}{16\pi^2\, E_1\, E_2}\,\delta(E_{cm}-E_1-E_2)$

$$= \int \frac{dp_1\, d\Omega\, p_1^2}{16\pi^2\, E_1 E_2}\,\delta\left(E_{cm}-\sqrt{m_1^2+p_1^2}-\sqrt{m_2^2+p_1^2}\right)$$

Recall $\displaystyle\int \delta(f(x))\, g(x)\, dx = \sum_k \frac{g(\alpha_k)}{|f'(\alpha_k)|}$

where α_k are zeroes of $f(x)$.

Solve $E_{cm} = \sqrt{p_1^2+m_1^2}+\sqrt{p_1^2+m_2^2}$ for p_1

$$E_{cm}-E_1 = E_2 \Rightarrow E_{cm}^2 - 2E_{CM}E_1 + E_1^2 = E_2^2$$

$$\Rightarrow E_{CM}^2 - 2E_{CM}\sqrt{p_1^2+m_1^2} + p_1^2+m_1^2 = p_1^2+m_2^2$$

$$\Rightarrow E_{CM}^2 + m_1^2 - m_2^2 = 2E_{CM}\sqrt{p_1^2+m_1^2}$$

$$\Rightarrow \left(\frac{E_{CM}}{2}+\frac{m_1^2-m_2^2}{2E_{CM}}\right)^2 - m_1^2 = p_1^2$$

$$p_1^2 = \left(\frac{E_{CM}}{2}+\frac{m_1^2-m_2^2}{2E_{CM}}+m_1\right)\left(\frac{E_{CM}}{2}+\frac{m_1^2-m_2^2}{2E_{CM}}-m_1\right)$$

$$\text{or } p_1^2 = \frac{E_{CM}^2}{4} + \frac{m_1^2-m_2^2}{2\cancel{E_{CM}}}\cdot\cancel{E_{CM}} + \frac{(m_1^2-m_2^2)^2}{4E_{CM}^2} - m_1^2$$

$$p_1^2 = \frac{E_{CM}^2}{4} - \frac{(m_1^2+m_2^2)}{2} + \frac{(m_1^2-m_2^2)^2}{4E_{CM}^2}$$

$$= \frac{E_{CM}^2}{4}\left(1 - \frac{2(m_1^2+m_2^2)}{E_{CM}^2} + \frac{(m_1^2-m_2^2)^2}{E_{CM}^4}\right)$$

$$p_1 = \frac{E_{CM}}{2}\left(1-\frac{2}{E_{CM}^2}(m_1^2+m_2^2)+\frac{(m_1^2-m_2^2)^2}{E_{CM}^4}\right)^{1/2}$$

$$f'(p_1) = \frac{-1}{\sqrt{m_1^2+p_1^2}} \cdot \frac{2p_1}{2} - \frac{1}{\sqrt{m_2^2+p_1^2}} \frac{2p_1}{2} = -\frac{p_1}{E_1} - \frac{p_1}{E_2}$$

$$\int d\Pi_2 = \int \frac{dp_1}{16\pi^2} p_1^2 \frac{d\Omega}{E_1 E_2} \, \delta(E_{CM} - E_1 - E_2)$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{p_1^{\cancel{2}}}{E_1 E_2} \cdot \frac{1}{\cancel{p_1}} \left(\frac{1}{E_1} + \frac{1}{E_2} \right)^{-1} \Bigg|_{p_1 \text{ from } \delta\text{-fcn.}}$$

$$= \int \frac{d\Omega}{16\pi^2} \cdot \frac{|\vec{p}_1|}{\cancel{E_1 E_2}} \frac{\cancel{E_1 E_2}}{(E_1+E_2)}$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{|\vec{p}_1|}{E_{CM}}$$

in COM frame, $E_{CM} = E_1 + E_2$

In general, we can use azimuthal symmetry to integrate $\int d\Omega = \int d\cos\theta \, d\phi = 2\pi \int d\cos\theta$

$$\int d\Pi_2 = \int \frac{d\cos\theta}{8\pi} \frac{|\vec{p}_1|}{E_{CM}}$$

Typically, the matrix element squared is depedent on spin correlations, so we cannot proceed with the $\cos\theta$ integral. Thus, for 2-to-2 scattering, we get

$$d\sigma = \int d\Omega \frac{1}{2E_A \, 2E_B} \frac{1}{|v_A - v_B|} \cdot \frac{|\vec{p}_1|}{16\pi^2 E_{CM}} \, |\mathcal{M}(p_A, p_B \to p_1, p_2)|^2$$

where energies, velocities, & momenta are in the COM frame.

We also can intuit the differential decay rate: "remove" particle B & assume particle A is at rest = COM frame by definition:

$$d\Gamma = \frac{1}{2m_A} \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f} \right) |\mathcal{M}(m_A \to \{p_f\})|^2 (2\pi)^4 \delta^{(4)}(p_A - \Sigma p_f)$$

Note: when final states include n identical particles, we have to divide by $n!$ to account for over counting the phase space.

We can also perform one manipulation of the flux factor:

$$\frac{1}{2E_A 2E_B}\frac{1}{|v_A - v_B|} = \frac{1}{4|p_A^z E_B - p_B^z E_A|} = \frac{1}{4|p_A^3 p_B^0 - p_B^3 p_A^0|}$$

$$= \frac{1}{4|\epsilon_{\mu 1 2 \nu}\, p_A^\mu p_B^\nu|}$$

Consider frame where particle A is at rest + particle B has momentum $\vec{p}_B = \underbrace{\sqrt{E_B^2 - m_B^2}}_{p_B^z}\,\hat{z}$

$\overset{B}{\bullet} \longrightarrow \;\bullet^{A}$

Then $E_A = m_A,\ p_A^z = 0$

(center-of-mass energy)^2 $\quad s = (p_A + p_B)^2 = m_A^2 + 2m_A E_B + m_B^2$

$$E_B = \frac{1}{2m_A}\left(s - m_A^2 - m_B^2\right)$$

$$p_B^z = \frac{1}{2m_A}\sqrt{(s - m_A^2 - m_B^2)^2 - 4m_A^2 m_B^2}$$

$$\Rightarrow \frac{1}{4|p_A^z E_B - p_B^z E_A|} = \left(2\lambda^{1/2}(s, m_A^2, m_B^2)\right)^{-1}$$

where $\lambda(x,y,z) = \left(x^2 + y^2 + z^2 - 2(xy + yz + zx)\right)$ is the Källén function. Note that $\lambda(x,y,0) = (x-y)^2$ and is symmetric for all three variables.

This gives the compact expression

$$d\sigma = \frac{1}{2\lambda^{1/2}(s, m_A^2, m_B^2)}\, d\Pi_n\, |\mathcal{M}|^2$$

(Since the flux factor is expressed using scalars, it is frame-independent, but should only calculate in COM frame.)

Finally, we address how to calculate matrix elements & the complications of the interacting theory.

We have

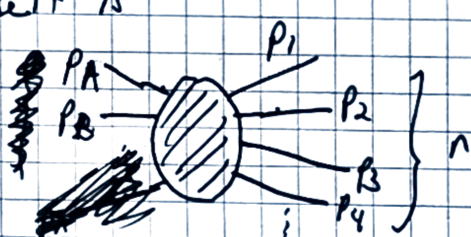
$$\langle \vec{p}_1 \vec{p}_2 \cdots | iT | \vec{p}_A \vec{p}_B \rangle = (2\pi)^4 \delta^{(4)}(p_A + p_B - \Sigma p_f) \, i\mathcal{M}(p_A, p_B \to \{p_f\})$$

The LHS is what we need to relate to free theory + particle states:

$$\langle p_1 p_2 \cdots | iT | p_A p_B \rangle \overset{?}{=} \lim_{T \to \infty(1-i\epsilon)} {}_0\langle p_1 p_2 \cdots | T\exp\left[-i\int_{-T}^{T} dt \, H_{int}(t)\right] | p_A p_B \rangle_0$$

where the states on the RHS are identified in the free theory.

The RHS itself is

[RHS] = 

(See P+S, eqn. 4.90) = $(2+n)$-pt. correlation function, i.e. fully ~~contracted~~ connected diagram with <u>external propagators removed</u>.

It is straightforward to extend the initial state to m possible particles.

The reason this is conceptually nontrivial is because there is no time where we can claim that the eigenstates of H are equivalent to the eigenstates of H_0.

We were only able to do this for the vacuum state,

$$|\Omega\rangle = \lim_{T \to \infty(1-i\epsilon)} \left(e^{-iE_0 T} \langle \Omega | 0 \rangle \right)^{-1} e^{-iHT} |0\rangle,$$

since the vacuum has lowest energy.

For particle states, however, we can only try to relate asymptotic states (which are one-particle states of the interacting theory) to free one-particle states via an unknown constant Z denoting the overlap

$$|p_1\rangle_0 \xrightarrow{t\to-\infty} Z_1^{1/2}\ |p_1\rangle_{in}$$

Intuitively, this factor is roughly $\langle\Omega|\phi(x)|\lambda_{\vec{p}}\rangle$, where $\lambda_{\vec{p}}$ is an eigenstate of momentum $\vec{p}$.

This is called the wavefunction renormalization constant, and reflects the changes in vacuum from free to interacting as well as particles.

The formal discussion of this effect is derived from the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula, which encapsulates the formal modifications of how a one-particle free state changes into new one-particle states, bound states, & multiparticle states in an interacting theory. (via the Källén-Lehmann spectral representation formula). [See P&S, section 7.1]

Quick intuition:

Consider two particles with an interaction.

Treat one particle as a central potential.

Given a smooth variation of second particle momenta & depending on sign of the potential term, we can get the first particle to escape or get captured

"free" int. particle — bound state

If we allow particles to be created by vacuum, then can also have multiparticle states.

Adopting the ansatz that asymptotic states are related by $Z_i^{1/2}$ factors to free non-interacting states, we have

[Position space]

$$i\mathcal{M}(p_1 \ldots p_m \to p_1' \ldots p_n')\,(2\pi)^4\,\delta^{(4)}\left(\Sigma p_1 \ldots p_m - \Sigma p_1' \ldots p_n'\right)$$

$$= \left(\Sigma \text{ fully connected, amputated Feynman diagrams with } p_1 \ldots p_m \text{ incoming } \& \; p_1' \ldots p_n' \text{ outgoing}\right) \cdot Z_1^{1/2} \ldots Z_m^{1/2} \cdot Z_{1'}^{1/2} \ldots Z_{n'}^{1/2}$$

Essentially, the reason for amputation is the one-particle version of time evolution between the free particle into the interacting one-particle state.

Next time: Amputation of diagrams, Feynman rules for ext. states, Feynman rules for fermions.