Lecture 13.
Quantum Field Theory.
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Last time: Interacting QFTs, rules for writing Lagrangians
Today: Interaction picture and time-ordered products

Having studied free theory for scalars & fermions, we have considered the time-dependence in Heisenberg picture:

Heisenberg: operators time-dep.

Schrödinger: states time-dep.

$$\phi_H(x) = \underset{\substack{\uparrow \\ \text{Heisenberg}}}{\phi_H}(t,\vec{x}) = e^{iHt}\underset{\substack{\uparrow \\ \text{Schrödinger}}}{\phi_S}(\vec{x})e^{-iHt}$$

So far, $\phi_H(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(e^{-ip\cdot x}a_{\vec{p}} + e^{+ip\cdot x}a^\dagger_{\vec{p}}\right)$

time dep. no time dep.

When we include interactions, such as

$$\mathcal{L} = \mathcal{L}_{free} - \frac{a}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4,$$ we would need to solve

$$(\Box + m^2)\phi(x) = -\frac{a}{2}\phi^2 - \frac{\lambda}{6}\phi^3,$$ but this is not known. In particular, getting appropriate $a_{\vec{p}}(t)$ + $a^\dagger_{\vec{p}}(t)$ is not tractable.

We want to solve

$$\begin{cases} \phi(t,\vec{x}) = e^{iHt}\phi(0,\vec{x})e^{-iHt} \\ \frac{\partial}{\partial t}\phi(t,\vec{x}) = i[H,\phi(t,\vec{x})] \end{cases}$$, with H as full Hamiltonian

We will adopt the Interaction picture, which is between the Heisenberg + Schrödinger pictures, and allows us to treat interactions perturbatively.

In the interaction picture, the time dependence from H_0 is put into operators & the time dependence from H_{int} is put into states.

We have $|\Psi(t)\rangle_I = e^{iH_0 t}|\Psi(t)\rangle_S = e^{iH_0 t}e^{-iHt}|\Psi\rangle_H$

$$\mathcal{O}_I(t) = e^{iH_0 t}\,\mathcal{O}_S\,e^{-iH_0 t},$$

and expect that for weak interactions, the most important time dependence is generated by H_0.

To get to the interaction picture from Heisenberg picture, fix $\phi_I(0,\vec{x}) = \phi_H(0,\vec{x})$ for ref. time $t_0 = 0$.

$$H = H_0^H + H_{int}^H = H_0^I + H_{int}^I$$

$$H_0^H = H_0^I \text{ @ } t=0, \quad H_{int}^H = H_{int}^I \text{ @ } t=0.$$

Then, define $\phi_I(t,\vec{x}) = e^{iH_0^I t}\,\phi_I(0,\vec{x})\,e^{-iH_0^I t}$

so $\frac{\partial}{\partial t}H_0^I = i[H_0^I, H_0^I] = 0$ is time-independent.

We have $\frac{\partial}{\partial t}\phi_I(t,\vec{x}) = i[H_0^I, \phi_I(t,\vec{x})]$

free Hamiltonian expressed in terms of ϕ_I fields

as generator of time evol.

States evolve as

$$|\Psi(t)\rangle_I = e^{iH_0 t}\,e^{-iHt}\,|\Psi\rangle_H$$

$$\frac{\partial|\Psi(t)\rangle_I}{\partial t} = iH_0|\Psi(t)\rangle_I - ie^{iH_0 t}He^{-iHt}|\Psi\rangle_H$$

$$= iH_0|\Psi(t)\rangle_I - ie^{iH_0 t}He^{-iH_0 t}|\Psi(t)\rangle_I$$

since $H = H_0 + H_{int}$,

time-ind. Schröd. Hamiltonian

$$= -ie^{iH_0 t}\left(H_{int}(0)\right)e^{-iH_0 t}|\Psi(t)\rangle_I$$

The time evolution of the state is driven by non-trivial H_{int}. Can check that the amplitudes in Heisenberg + Interaction pictures are equivalent.

The key advantage of the interaction picture is that H_{int} can be expressed using free creation + annihilation operators with same time dependence as free field operators. So instead of time evolution in the usual Heisenberg fashion,

$$\phi_H(t,\vec{x}) = e^{iH(t-t_0)} \phi(t_0,\vec{x}) e^{-iH(t-t_0)}$$

we have $\phi_I(t,\vec{x}) = e^{iH_0(t-t_0)} \phi(t_0,\vec{x}) e^{-iH_0(t-t_0)}$

Then, given $\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(e^{-ip\cdot x} a_{\vec{p}} + e^{ip\cdot x} a^\dagger_{\vec{p}}\right)\Big|_{\substack{x^0 = t-t_0 \\ p^0 = E_{\vec{p}}}}$

we have $\phi_H(x) = e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \underset{\overset{\parallel}{(t,\vec{x})}}{\phi_I(x)} e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$

$$= U^\dagger(t,t_0)\, \phi_I(t,\vec{x})\, U(t,t_0)$$

for $U(t,t_0)$ = unitary time evolution operator from t_0 to t.

$$= e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

Note that we cannot generally simplify $U(t,t_0) = e^{i(H_0-H)(t-t_0)}$ because of Baker, Campbell, Hausdorff.

At $t=t_0$, $U(t_0,t_0) = 1$.

States evolve with $U(t,t_0)$:

$$|\Psi(t)\rangle_I = U(t,t_0)\,|\Psi(t_0)\rangle_I = U(t,t_0)\,|\Psi(t_0)\rangle_H$$

time-ind. Heisenberg state

Now, $\frac{\partial}{\partial t} U(t,t_0) = iH_0\, e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$

$$+ e^{iH_0(t-t_0)} (-iH) e^{-iH(t-t_0)}$$

$$= iH_0\, U(t,t_0)$$

$$- i\, e^{iH_0(t-t_0)} H e^{-iH_0(t-t_0)} U(t,t_0)$$

$$= -i\, e^{iH_0(t-t_0)} H^I_{int}(t_0) e^{-iH_0(t-t_0)} U(t,t_0)$$

$$i\frac{\partial}{\partial t} U(t,t_0) = H^I_{int}(t)\, U(t,t_0)$$

Now, we can solve this differential equation by an appropriate exponential, integrated over time. The main complication is that $H^I_{int}(t)$ is an operator, so fields in the interaction picture at different times do not generally commute:

$$[\phi^I_{int}(t_1), \phi^I_{int}(t_2)] \neq 0 \quad \text{for } t_1 \neq t_2.$$

Instead of the naive solution $U(t,t_0) \sim \exp\left(-i\int_{t_0}^{t} dt'\, H^I_{int}(t')\right)$, we have the Dyson series solution:

$$U(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt_1'\, H^I_{int}(t_1')$$
$$+ (-i)^2 \int_{t_0}^{t} dt_1' \int_{t_0}^{t_1'} dt_2'\, H^I_{int}(t_1')\, H^I_{int}(t_2') + \ldots$$

Can verify by differentiating w.r.t. to time.

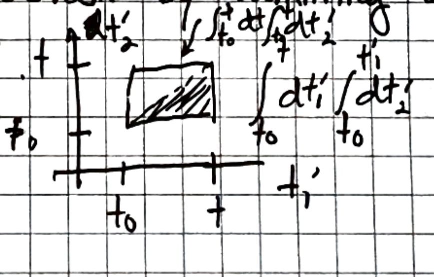
By using T = time ordering symbol, we can compactify the notation:

$$\int_{t_0}^{t} dt_1' \int_{t_0}^{t_1'} dt_2'\, H^I_{int}(t_1')\, H^I_{int}(t_2')$$
$$= \frac{1}{2} \int_{t_0}^{t} dt_1' \int_{t_0}^{t} dt_2'\, T\left\{ H^I_{int}(t_1')\, H^I_{int}(t_2') \right\}$$

and

$$\int_{t_0}^{t} dt_1' \int_{t_0}^{t_1'} dt_2' \ldots \int_{t_0}^{t_{n-1}'} dt_n'\, H^I_{int}(t_1') \ldots H^I_{int}(t_n')$$
$$= \frac{1}{n!} \int_{t_0}^{t} dt_1'\, dt_2' \ldots dt_n'\, T\left\{ H^I_{int}(t_1') \ldots H^I_{int}(t_n') \right\}$$

In effect, we always want the lower triangular prism shape, which we can obtain by summing over the n-dimensional cube & dividing by $n!$.



See fig. 4.1 of P&S.

Thus, the unitary time evolution operator is

$$U(t,t_0) = T\left\{ \exp\left[-i\int_{t_0}^{t} dt' H_{int}^{I}(t')\right]\right\}$$

Recall $T\{\dots\}$ places all objects in time-ordering, and includes an extra -1 when fermionic operators are interchanged. (See P+S, eqn. 3.121)

We can further generalize $U(t,t') = T\left\{\exp\left[-i\int_{t'}^{t} dt'' H_{int}^{I}(t'')\right]\right\}$ from $U(t,t_0)\,U^{\dagger}(t',t_0) = U(t,t')$.

Remark:
Note that subsequent terms in $U(t,t')$ are higher powers of H_{int}^{I}, which necessarily involve higher powers of our perturbative coupling(s). Hence, we can interpret the Dyson series as a perturbative solution & truncate at a given order in our coupling expansion.

Next time:

Interacting vacuum vs. free vacuum

Correlation fcns. & S-matrix.