Lecture 12.
Quantum Field Theory,
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Last time: Discrete symmetries, CPT
Today: Interacting quantum field theories.

We have now addressed 2 of the basic free fields we will use in this course: scalars + ~~...~~ fermions. The last type of free field needed is a vector field, but we will postpone their quantization until we discuss gauge invariance.

Will now pivot to interacting field theories.
Now, in principle, it seems like the set of possible interactions we can consider is infinite. For a given ~~...~~ set of fields, we only need to follow the rules of a Lagrangian term: Recipe for building a Lagrangian.

1. Lagrangian must be Hermitian.
2. Each Lagrangian term must be Lorentz-invariant.
3. Every term must be mass dimension-4. Typically, the fields are multiplied together and then an appropriate coupling constant with the correct mass dimension is multiplied as a prefactor.

In 4D QFTs, scalar fields have mass dimension 1, fermions have mass dimension 3/2, and vector fields are mass dimension 1.

Aside: Results from $\mathcal{L} \supset \frac{1}{2}(\partial_\mu\phi)^2 + \bar{\Psi} i \not{\partial} \Psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$

4. All terms must be invariant under assigned internal (gauge or global) symmetry transformations.
   - The most general transformation of a field under a symmetry transformation involves transforming spacetime as well as the field itself. It seems then that we would need to include additional spacetime-acting generators to the Lorentz algebra. But, it can be proven that the Lorentz (Poincaré) generators saturate the possible group actions and no non-trivial generators acting on spacetime can be added. Hence, any symmetry transformation acting on a field must act on the field + spacetime separately (must factorize). This is the essence of the Coleman-Mandula theorem.

Aside: The Haag-Kopuszański-Sohnius theorem provided one extension to the Coleman-Mandula theorem, where the generators of spacetime Poincaré symmetry can be extended by new generators obeying anticommutation relations. This necessarily requires spacetime to be augmented by fermionic coordinates labeled by Grassmann numbers, leading to supersymmetry.

Thus, we have already addressed spacetime symmetry behavior of scalars & fermions & vectors. We now have possible phase rotations:

Global phase rotation:

$\phi(x) \to \phi' = e^{iq\alpha}\phi(x)$, α is independent of x
q is charge.

Gauge phase rotation:

$$\phi(x) \to \phi'(x) = e^{iq\alpha(x)}\phi(x).$$

The first will be a $U(1)$ global symmetry,
($U(1)$ = unitary group with one generator,
equiv. to $e^{i\alpha\mathbb{1}}$)
α: infinitesimal ~~gen~~ param.
$\mathbb{1}$: generator

and the second is a $U(1)$ gauge symmetry.
The main complication of gauge symmetry is that kinetic terms $\left[\frac{1}{2}(\partial_\mu\phi)^2 \text{ or } \bar{\Psi} i\not{\partial}\Psi\right]$ are no longer invariant, because the derivative acts non-trivially on the phase parameter. Will need to add a connection field to absorb the phase change between different spacetime points $x \to x+\delta x$, which becomes a gauge field coupling inside a covariant derivative, $\partial_\mu \to D_\mu$.

*Minimal coupling: All vector interactions are implemented via covariant derivatives.

Aside: We can also have more complicated global + gauge symmetry transformations. Given that they are not discrete transformations, then the sym. groups must be continuously connected to the identity element by infinitesimal params. This leads to a complete classification of Lie algebras, which is (surprisingly) a finite set of possibilities

These are best understood by detouring to matrix-valued fields, but this is beyond the scope of these lectures + is instead reserved for QFT 2. We will also do non-Abelian gauge theory there, which allows for gauge field self-interactions.

The global $U(1)$ phase rotation we have already seen when studying the complex scalar field. The gauge $U(1)$ theory is most appropriately taught in the context of quantum electrodynamics ($U(1)$ gauge theory with fermions) or scalar quantum electrodynamics ($U(1)$ gauge theory with scalar fields).

QED is the main interacting QFT we study in this course, but we will also see $\lambda\phi^4$ theory and Yukawa theory, $y\phi\bar{\psi}\psi$.

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5. All terms invariant under given symmetry rules should be written down. In quantum physics, what is not forbidden is compulsory. In other words, if a process or interaction does not exist, then it can generally be traced to violation of a selection rule on the quantum states. If such a selection rule does not exist, then the process will occur via quantum effects = from explicit, (perturbative) loop corrections.

6. We can also make a leap to characterize super-renormalizeable, renormalizeable, and non-renormalizeable Lagrangian terms.

Super-renormalizable: canonical dimension of fields is less than 4. Coupling constant has positive mass dimension.

Renormalizable: canonical dimension of fields is equal to 4. Coupling constant is dimensionless.

Non-renormalizeable: canonical dimension of fields is greater than 4. Coupling constant has negative mass dimension.

Example: $m^2|\phi|^2$ — super-renorm.

$m\bar{\psi}\psi$ - super-renorm.

$\lambda\phi^4$ - renorm.

$\frac{c}{\Lambda}\phi^5$ - non-renorm.

$\frac{c}{\Lambda^2}\bar{\psi}\psi\bar{\psi}\psi$ - non-renorm.

With these rules, we can actually list the finite set of possible interactions (i.e. Lagrangian terms with 2 fields are kinetic terms or mass terms, while Lagrangian terms with 3 or more fields are interactions, leading to non-linearities in EOMs).

$$a\phi^3, \quad \lambda\phi^4, \quad y\phi\bar{\psi}\psi, \quad iy'\phi\bar{\psi}\gamma^5\psi,$$
$$e\bar{\psi}\gamma^\mu A_\mu\psi, \quad ie(-\phi^*\overleftarrow{\partial}_\mu\phi + \phi^*\partial_\mu\phi)A^\mu$$
$$e'\bar{\psi}\gamma^\mu\gamma^5 A_\mu\psi, \quad (b\phi|\phi|^2 + h.c.)$$

Surprisingly, (after we allow for non-Abelian gauge fields) the only interactions seen in Nature all fall into one of these types, and even not all of these have been observed / are predicted to be absent in the SM.

The restriction to building only renormalizeable Lagrangians (i.e. those without non-ren. terms) is not only motivated phenomenologically, but also from a formal perspective.

When we add an interaction to the Lagrangian, the quantum states are, in principle, dramatically changed as a result. The eigenstates of the free Hamiltonian are not, in principle, related to the eigenstates of an interacting Hamiltonian. (Hilbert spaces do not coincide.)

Ex: double well potential.

We only had a few exactly solvable theories in QM, but we have no solvable QFTs in 4D.
So, we will always adopt a perturbative approach, where interaction couplings are treated as small parameters + leading effects are calculated.
But, it is still a very strong assumption that the states of the free theory are "smoothly connected" to the states of the interacting theory as the coupling is turned on.

This leads us to the interaction picture of
treating time evolution of free fields exactly +
time evolution of interactions perturbatively.

Next time: Time-ordered products

+ justification for restriction to ren. $\mathcal{L}$.