08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie Quantum Field Theory I

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Homework set 8

Due June 14, 2021; e-mail (photo or scan) to yu001@uni-mainz.de by start of lecture.
Please note how long it took you to solve each problem.

8-1, 35 pts. *The Optical Theorem.* (*Aside:* There is a central result in quantum field theory about loop diagrams and their relation to tree-level diagrams. While we do not yet have the technical expertise to evaluate or calculate loop diagrams, we can nonetheless observe the gist of the optical theorem already.)

A, 5 pts. Show that the unitarity of the S matrix leads to the requirement $i(T - T^\dagger) = -T^\dagger T$, where $S = 1 + iT$ (these are all infinite-dimensional matrices in the Fock space of the particle states).

B, 15 pts. Using the defining relation between transition matrices and matrix elements $\mathcal{M}$, reexpress the above equation as an amplitude between the *same* initial and final state $|f\rangle = |i\rangle$. For the right hand side, insert a complete set of multi-particle states

$$1 = \sum_{n=1}^{\infty} \prod_{i=1}^{n} \int \frac{d^3k_i}{(2\pi)^3 2k_i^0} |k_1 \ldots k_n\rangle\langle k_1 \ldots k_n| \qquad (1)$$

Elminate all T's from the answer and rewrite it in terms of the matrix elements $\mathcal{M}(i \to i)$ and $\mathcal{M}(i \to k)$ (where k is anything, by virtue of the complete set of states integration).

C, 15 pts. Write a paragraph interpreting this result as the *optical theorem*. You may find it useful to consider the powers of coupling constants that must enter on each side of the equation. In particular, one side of the equation can be considered as the imaginary part of the forward-scattering amplitude $\mathcal{M}(i \to i)$, while the other side is related to the cross section of $i \to$ anything. If needed, take a concrete case such as ϕ^4 theory. (*Aside:* A more extensive discussion of the optical theorem can be found in Section 7.3 of Peskin and Schroeder, or Section 24.1 of Schwartz.)

8-2, 25 pts. *Practice with drawing Feynman diagrams.* Draw all fully connected Feynman diagrams contributing to the $\phi\phi \to \phi\phi$ transition amplitude $\mathcal{M}(p_1p_2 \to p_3p_4)$ at $\mathcal{O}(\lambda)$, $\mathcal{O}(\lambda^2)$, and $\mathcal{O}(\lambda^3)$ in $\lambda\phi^4$ theory. You may ignore any diagrams with non-amputated external legs.

8-3, 40 pts. *Scalar decay width into two scalars.* Consider the Lagrangian for two distinct real scalar fields ϕ and σ,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \sigma\right)^2 + \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}M^2\sigma^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}\mu\sigma\phi^2 \ , \tag{2}$$

where μ is a coupling constant with mass dimension 1. We will assume $M > 2m$, which leads to a possible decay of $\sigma \to \phi\phi$.

A, 20 pts. Draw all fully connected Feynman diagrams that contribute to the decay $\sigma \to \phi\phi$, up to $\mathcal{O}(\mu^3)$.

B, 20 pts. The differential decay rate for a given σ particle with momentum p_1 to decay into two ϕ particles with momenta p_2 and p_3 is

$$d\Gamma(\sigma \to \phi\phi) = \frac{1}{2}\frac{1}{2M}(2\pi)^4\delta^{(4)}(p_1 - p_2 - p_3)\frac{d^3p_2}{(2\pi)^3 2p_2^0}\frac{d^3p_3}{(2\pi)^3 2p_3^0}|\mathcal{M}(\sigma \to \phi\phi)|^2 \ , \tag{3}$$

where we have written an extra $\frac{1}{2}$ pre-factor to account for the fact that the final states are identical, so integrating over the two particle phase space is redundant by a factor of 2. Given that the matrix element $\mathcal{M} = \mu$ (at lowest order in μ) does not depend on the momenta of the particles, integrate this expression over the final state phase space to get the decay rate $\Gamma(\sigma \to \phi\phi)$.