08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie Quantum Field Theory I

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Homework set 6

Due May 31, 2021; e-mail (photo or scan) to yu001@uni-mainz.de by start of lecture.
Please note how long it took you to solve each problem.

6-1, 15 pts. Construct and verify (by explicit calculation) the set of five possible Hermitian Dirac fermion bilinears. *Hint: Recall the Lorentz-invariant or Lorentz-covariant bilinears are*

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^5\psi, \quad \bar{\psi}\gamma^\mu\psi, \bar{\psi}\gamma^\mu\gamma^5\psi, \bar{\psi}\sigma^{\mu\nu}\psi \ . \tag{1}$$

If the bilinear listed above is not already Hermitian, you should be able to obtain a Hermitian bilinear by multiplying by "i". Hint 2: For the vector and axial-vector bilinears, you should first prove the identity

$$(\gamma^\mu)^\dagger \gamma^0 = \gamma^0\gamma^\mu \ . \tag{2}$$

6-2, 60 pts. The library of possible interactions with scalar, fermion, and vector fields. In this exercise, we will enumerate all possible Lagrangian interaction terms that occur in a perturbative, renormalizable treatment of QFT (except for non-Abelian gauge fields).

A, 10 pts. Write the canonical kinetic terms of a real free scalar field, a complex free scalar field, a Dirac fermion, and a free Abelian vector field. Derive the mass dimension of scalar, fermion, and vector fields in D-dimensional quantum field theory. What is the mass dimension when $D = 4$? *Note: Refer to Lecture 12 as needed. The mass dimension is also known as the scaling dimension or the canonical dimension.*

B, 5 pts. For the real scalar, complex scalar, and the Dirac fermion from part (A), write an appropriate Lagrangian mass term. *Remark: You should note that mass terms are always scalar contractions of field bilinears.*

C, 30 pts. List all of the possible dimension-3 and dimension-4 interaction Lagrangian terms built out of real scalars, fermions, vectors, and partial derivatives (and gamma matrices). For the dimension-3 terms, you will need to multiply the fields by a coupling constant of mass dimension 1 to ensure they have overall mass dimension 4. **Note: You do not need to consider any terms with more than one vector field.**

D, 15 pts. How do the Lagrangian terms from part (C) change if the scalar field is complex instead of real?

Extra credit, E, 10 pts. Ignoring minimal coupling, write all possible renormalizable Lagrangian terms with two or more vector fields. You should argue that one of these terms is redundant with the normal kinetic term of the vector field. *Note: "Renormalizable" means that the coupling constant has either positive or null mass dimension.*

6-3, 25 pts. Multiple scalar fields and the Linear sigma model (and flavor/horizontal symmetries). Recall that the Lagrangian for a single real scalar field is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - V(\phi) \ . \tag{3}$$

For a free field, $V(\phi) = \frac{1}{2}m^2\phi^2$. For the purposes of this exercise, we will again only consider renormalizable Lagrangian terms (*i.e.* those up to fourth power in scalar field). *We will also ignore any "tadpole" terms that are linear in a single scalar field.*

A, 5 pts. What two additional interaction terms can you include in $V(\phi)$?

B, 10 pts. Now consider two separate real scalar fields, ϕ_1 and ϕ_2, with separate masses. Assume that their kinetic terms and mass terms are always unmixed, so we do not have $\partial_\mu\phi_1\partial^\mu\phi_2$, for example. What are all of the cubic and quartic interaction terms you can write?

C, 10 pts. Generalize the answer in part (B) to N real scalar fields, keeping the kinetic and mass terms unmixed. You can use the notation of a column vector $\Phi = (\phi_1, \phi_2, \ldots \phi_N)^T$. If you impose an $O(N)$ symmetry invariance on the Lagrangian, what couplings relations are required? Specifically, require the Lagrangian to be invariant under the transformation $\Phi \to \Phi'_i = R_{ij}\Phi_j$, where R is an orthogonal matrix ($R^TR = RR^T = 1_{N\times N}$).