08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie Quantum Field Theory I

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Homework set 4

Due May 10, 2021; e-mail (photo or scan) to yu001@uni-mainz.de by start of lecture.
Please note how long it took you to solve each problem.

4-1, 12 pts. Spinor space. Use the explicit form of the Pauli matrices to compute the eigenvalues of $p\cdot\sigma$ and $p\cdot\bar{\sigma}$, where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$. Show that for an on-shell particle ($p^2 = m^2$, $p^0 > 0$) that the eigenvalues are always positive. Obtain the explicit form of the spinor $u_s(p)$ for a particle moving in the $+\hat{x}$ direction.

4-2, 12 pts. Gordon identity (problem 3.2 of Peskin and Schroeder). Derive the Gordon identity,

$$\bar{u}_r(p')\gamma^\mu u_s(p) = \bar{u}_r(p')\left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u_s(p) \tag{1}$$

for $q = p' - p$. You can use the constraint $(\not{p} - m)u_s(p) = 0$ on the spinor from the Dirac equation.

4-3, 12 pts. Lorentz transformations of bilinear spinor contractions. Given that $\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu$ for the Dirac matrices γ^μ and defining $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$, show that under a Lorentz transformation

A, 6 pts. $\bar{\psi}\gamma_5\psi \to \det(\Lambda)\bar{\psi}\gamma_5\psi$,

B, 6 pts. $\bar{\psi}\gamma^\mu\gamma_5\psi \to \det(\Lambda)\Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\gamma_5\psi$.

[Aside: The first fermion bilinear is a pseudoscalar contraction, and the second fermion bilinear is a pseudovector contraction. In general, $\det(\Lambda) = +1$ for continuous Lorentz transformations, but we can also have $\det(\Lambda) = -1$ if we perform a discrete Lorentz transformation such as a spatial reflection (known as parity).]

4-4, 24 pts. Practice with Dirac algebra (part 1). Evaluate the following products of γ matrices with contracted indices. (Hint: See equation 5.9 of Peskin and Schroeder for some solutions.)

A, 4 pts. $\gamma^\mu\gamma_\mu$

B, 4 pts. $\gamma^\mu\gamma^\nu\gamma_\mu$

C, 4 pts. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu$

D, 4 pts. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu$

E, 4 pts. $(\gamma_5)^2$

F, 4 pts. Define

$$P_L = \frac{1_{4\times4} - \gamma_5}{2} \tag{2}$$

$$P_R = \frac{1_{4\times4} + \gamma_5}{2} \ . \tag{3}$$

Show that $(P_L)^2 = P_L$, $(P_R)^2 = P_R$, and $P_L P_R = 0$.

4-5, 40 pts. Majorana fermions. We start with the Weyl equation,

$$i\bar{\sigma}\cdot\partial\chi = 0 \ , \tag{4}$$

which is a relativistic equation for a massless fermion field that transforms as the upper two-component spinor ψ_L of the Dirac spinor. We will label the components of the field by a, $\chi_a(x)$, $a = 1, 2$.

A, 8 pts. Show that the massive Majorana field equation,

$$i\bar{\sigma}\cdot\partial\chi - im\sigma^2\chi^* = 0 \tag{5}$$

is relativistically invariant and that it implies the Klein-Gordon equation. Here, σ^2 is the second Pauli matrix.

B, 16 pts. To consider a Lagrangian that gives the Majorana field equation from variations of χ, we need to consider $\chi(x)$ as a classical Grassmann-valued field. Grassmann numbers are *anti-commuting* numbers, which means that $\alpha\beta = -\beta\alpha$ for any Grassmann numbers α and β. Note also that $\alpha^2 = 0$. We define the complex conjugate of a product of Grassmann numbers as $(\alpha\beta)^* = \beta^*\alpha^* = -\alpha^*\beta^*$. Show that the classical action

$$S = \int d^4x \left[\chi^\dagger i\bar{\sigma}\cdot\partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right] \tag{6}$$

is real (i.e. $S^* = S$), where $\chi^\dagger = (\chi^*)^T$. Also show that varying the action with respect to χ and χ^* (treating them as independent dynamical variables) gives the Majorana equation.

C, 16 pts. Quantize the Majorana theory by promoting $\chi(x)$ to a quantum field satisfying the canonical anticommutation relation,

$$\left\{\chi_a(\vec{x},t), \chi_b^\dagger(\vec{y},t)\right\} = \delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}). \tag{7}$$

In other words, explicitly construct a Hermitian Hamiltonian and diagonalize it in terms of a set of creation and annihilation operators. (*Hint:* Compare $\chi(x)$ to the upper components of the quantized Dirac field.)