08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie
Quantum Field Theory I

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Homework set 3

Due May 3, 2021; e-mail (photo or scan) to yu001@uni-mainz.de by start of lecture.
Please note how long it took you to solve each problem.

3-1, 60 pts. Complex scalar fields, counting degrees of freedom and charge conservation (follows Problem 2.2 of Peskin and Schroeder). We consider a complex scalar field,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(e^{-ip\cdot x} a_{\vec{p}} + e^{ip\cdot x} b^\dagger_{\vec{p}} \right)\Big|_{p^0=E_{\vec{p}}} \tag{1}$$

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(e^{-ip\cdot x} b_{\vec{p}} + e^{ip\cdot x} a^\dagger_{\vec{p}} \right)\Big|_{p^0=E_{\vec{p}}} \tag{2}$$

$$\pi(x) = \dot{\phi}^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{E_{\vec{p}}}{2}} \left(e^{-ip\cdot x} b_{\vec{p}} - e^{ip\cdot x} a^\dagger_{\vec{p}} \right)\Big|_{p^0=E_{\vec{p}}} \tag{3}$$

$$\pi^\dagger(x) = \dot{\phi}(x) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{E_{\vec{p}}}{2}} \left(e^{-ip\cdot x} a_{\vec{p}} - e^{ip\cdot x} b^\dagger_{\vec{p}} \right)\Big|_{p^0=E_{\vec{p}}} \tag{4}$$

and the action

$$S = \int d^4x \left(\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 |\phi|^2 \right) . \tag{5}$$

A, 10 pts. Justify (or recalculate) the expressions for $\pi(x)$ and $\pi^\dagger(x)$ as conjugate momentum densities. Which π, $\pi^\dagger$ corresponds to which ϕ, $\phi^\dagger$ field?

B, 10 pts. Write the canonical, equal-time commutation relations between $\phi(x)$, $\phi^\dagger(x)$, $\pi(x)$ and $\pi^\dagger(x)$. (It is easiest if you consider $\phi(x)$ and $\phi^\dagger$ as *independent* dynamical variables.)

C, 10 pts. Show that the Hamiltonian for the theory is

$$H = \int d^3x \left(\pi^\dagger \pi + |\nabla \phi|^2 + m^2 \phi^\dagger \phi \right) \tag{6}$$

D, 10 pts. Verify that the complex scalar Lagrangian corresponds to two distinct real fields with equal masses. (It is sufficient to decompose the complex scalar field into real and imaginary components and rewrite the Lagrangian using these component fields: $\phi = \frac{1}{\sqrt{2}}(\phi_R + i\phi_I)$.) Can you justify the factor of $\frac{1}{\sqrt{2}}$?

E, 10 pts. If you perform a transformation of ϕ by $e^{i\alpha}$ (and a corresponding rotation of $\phi^\dagger$), is the action invariant? Derive the corresponding Noether current.

F, 10 pts. Show that the conserved charge from part (E) is defined by

$$Q = \int d^3x \frac{i}{2} \left(\phi^\dagger \pi^\dagger - \pi \phi \right) \,, \tag{7}$$

and rewrite this charge using creation and annihilation operators.

3-2, 40 pts. The Lorentz group. Read section 3.1 of Peskin and Schroeder (and refer to Lecture 2 notes as needed). The generators of the Lorentz group $J^{\mu\nu}$ obey the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = i \left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho} \right) \,. \tag{8}$$

We can furnish specific representations by assigning $J^{\mu\nu}$ to different mathematical objects that obey the above commutation relations.

A, 6 pts. How many generators are in the Lorentz group (for 3+1 spacetime dimensions)? What Lorentz transformations do they correspond to?

B, 10 pts. Demonstrate that $J^{\mu\nu} = L^{\mu\nu}$, with $L^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$, is a faithful representation by verifying that $L^{\mu\nu}$ obeys the commutation relations.

C, 10 pts. Demonstrate that $J^{\mu\nu} = \mathcal{J}^{\mu\nu}$, with $(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right)$ is also a faithful representation. This is known as the vector representation, since the corresponding 4×4 matrices act on 4-vectors.

D, 14 pts. For the vector representation, evaluate the explicit form of the transformation matrix $U = \exp\left(-\frac{i}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu} \right)$ for the following special cases:

i, 7 pts. $\omega_{13} = -\omega_{31} = \theta$, $\omega_{\mu\nu} = 0$ otherwise. Which axis of rotation does this correspond to?

ii, 7 pts. $\omega_{03} = -\omega_{30} = \beta$, $\omega_{\mu\nu} = 0$ otherwise. What boost does this correspond to?