08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie Quantum Field Theory I

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Homework set 12

Due July 12, 2021; e-mail (photo or scan) to yu001@uni-mainz.de by start of lecture.
Please note how long it took you to solve each problem.

12-1, 20 pts. *QED, soft photons, bremsstrahlung, and the electron vertex function.* Building from your reading from Section 6.1 last week, now read Section 6.4 of Peskin and Schroeder. Explain how the infrared divergence in the electron vertex function and the infrared divergence of bremsstrahlung are matched and cancel in any physical observable. *Aside: The general theorem of cancellation of infrared divergences is known as the Kinoshita-Lee-Nauenberg (KLN) theorem, and is a central result of quantum field theory.*

12-2, 60 pts. *Dimensional regularization.* The central mathematical tool to handle divergences in loop integrals in quantum field theory is the regularization prescription known as "dimensional regularization" (for which 't Hooft and Veltman won the Nobel prize in 1999). The key idea of dimensional regularization is that integrals of virtual loop momenta are always convergent if the number of spacetime dimensions d is small enough. Then, by analytic continuation, any observable quantity should be well-defined as $d \to 4$. We want to derive the master formula

$$I_d(a,b;\Delta) = \int \frac{d^d l_E}{(2\pi)^d} \frac{\left(l_E^2\right)^a}{\left(l_E^2+\Delta\right)^b} = \frac{\Delta^{d/2+a-b}}{(4\pi)^{d/2}} \frac{\Gamma(\frac{d}{2}+a)\Gamma(b-a-\frac{d}{2})}{\Gamma(\frac{d}{2})\Gamma(b)} \ , \qquad (1)$$

where $l_E^2 = \sum_{i=1}^d l_i^2$ is the square of the length of the d-dimensional momentum vector written in d-dimensional Euclidean space (after performing a Wick rotation from Minkowski space). Assume that $b > a + d/2$ and $\Delta \neq 0$, so the integral converges as $l_E^2 \to \infty$ and $l_E^2 \to 0$.

A, 20 pts. Factorize the d-dimensional integration measure as

$$d^d l_E = d\Omega_d \ (l_E^2)^{d/2-1} \ \frac{d(l_E^2)}{2} \ , \qquad (2)$$

where $d\Omega_d$ is the integration measure for the d angular coordinates of the sphere S_d and l_E^2 with $0 \leq l_E^2 < \infty$ is the square of the radial coordinate. Use the fact that $\sqrt{\pi} = \int_{-\infty}^{\infty} dx\ e^{-x^2}$ to express $\pi^{d/2}$ as an integral over d-dimensional Euclidean space, and evaluate the radial integral using the definition

$$\Gamma(z) = \int_0^\infty dt\ t^{z-1}\ e^{-t} \tag{3}$$

of the Γ-function. Calculate the surface $\int d\Omega_d$ of the sphere S_d. *Note: Recall that for integer z, $\Gamma(z = n) = (n-1)!$, and in general, $\Gamma(1+z) = z\Gamma(z)$. Hint: See pages 249-250 of Peskin and Schroeder for tips about performing the calculation.*

B, 20 pts. Evaluate the integral $I_d(a, b; \Delta)$ using the substitution $l_E^2/\Delta = (1-x)/x$ and the definition of the Euler β-function,

$$B(z, w) = \int_0^1 dx\ x^{z-1}\ (1-x)^{w-1} = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{4}$$

and derive the master formula.

C, 20 pts. Now, by analytic continuation, we can consider the master formula as the **definition** of $I_d(a, b; \Delta)$ for *arbitrary* values of a, b, and the number of (Euclidean) spacetime dimensions d. This is because the $\Gamma(z)$ function is analytic in the complex z plane except for isolated singularities at $z = 0, -1, -2, \ldots$. Verify that the master formula reproduces the results in equations 7.85 and 7.86 of Peskin and Schroeder:

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}}, \tag{5}$$

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{l_E^2}{(l_E^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}, \tag{6}$$

12-3, 20 pts. An alternative regularization prescription for divergent integrals is *Paulli-Villars regularization*, where a (fictitious) heavy particle is introduced to artificially enhance the convergence of the integral at infinity. After reading Sections 6.3 and 7.5 of Peskin and Schroeder, explain the correspondence between the number of spacetime dimensions in dim. reg. and the fictitious heavy particle mass in Pauli-Villars regularization.

12-4, 15 pts. *Extra credit* A key prediction of renormalization in quantum field theory is the fact that coupling constants *run*, i.e. they change as a function of the energy scale at which they are probed. In QED, the running coupling at 1-loop order is

$$e_R^2(Q^2) = e_R^2 \left[1 + \frac{e_R^2}{12\pi^2} \log\frac{Q^2}{m^2} + \mathcal{O}(e_R^4)\right], \tag{7}$$

where Q^2 is the energy scale squared where the coupling is probed, and the e_R on the RHS is interpreted as the *measured/extracted value* of the coupling at the scale

$Q^2 = m^2$. The β-function then characterizes how the coupling constant e_R changes with the energy scale Q:

$$\beta(e_R) \equiv Q \frac{de_R}{dQ} \ . \tag{8}$$

A, 5 pts. What is the β function of QED at 1-loop order?

B, 5 pts. What is the power counting for the leading 1-loop order correction in the β-function?

C, 5 pts. Given the sign of the β function, does the coupling increase or decrease as $Q \to \infty$? *Note: Theories where the coupling weakens in the ultraviolet scale are known as asymptotically free, while theories where the coupling strengthens in the ultraviolet are infrared free.*