08.128.165 Theorie 6a, Relativistische Quantenfeldtheorie Quantum Field Theory I

Felix Yu and Alexey Kivel, Julien Laux

Homework set 11

Due July 5, 2021; e-mail (photo or scan) to yu001@uni-mainz.de by start of lecture.
Please note how long it took you to solve each problem.

11-1, 15 pts. In general, the trace of an even number of gamma matrices is non-vanishing, and depending on how many internal fermion propagators and vertices belong to a continuous fermion line in a Feynman diagram, the number of gamma matrices can become arbitrarily large. In practice, the closed form identities of traces of gamma matrices become unwieldy when the number of gamma matrices goes beyond four, but based on problem 5-1, you should be able to understand how to generalize the trace identities to 6, 8, or further even numbers of gamma matrices. Explain the procedure for how you would derive the trace of an even number of gamma matrices. For reference,

$$\begin{aligned}\mathrm{Tr}\,[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta] = 4\,(& g_{\sigma\rho}g_{\alpha\nu}g_{\beta\mu} - g_{\sigma\nu}g_{\alpha\rho}g_{\beta\mu} - g_{\sigma\rho}g_{\alpha\mu}g_{\beta\nu} \\ & + g_{\sigma\mu}g_{\alpha\rho}g_{\beta\nu} + g_{\sigma\nu}g_{\alpha\mu}g_{\beta\rho} - g_{\sigma\mu}g_{\alpha\nu}g_{\beta\rho} + g_{\sigma\rho}g_{\alpha\beta}g_{\mu\nu} - g_{\sigma\beta}g_{\alpha\rho}g_{\mu\nu} + g_{\sigma\alpha}g_{\beta\rho}g_{\mu\nu} \\ & - g_{\sigma\nu}g_{\alpha\beta}g_{\mu\rho} + g_{\sigma\beta}g_{\alpha\nu}g_{\mu\rho} - g_{\sigma\alpha}g_{\beta\nu}g_{\mu\rho} + g_{\sigma\mu}g_{\alpha\beta}g_{\nu\rho} - g_{\sigma\beta}g_{\alpha\mu}g_{\nu\rho} + g_{\sigma\alpha}g_{\beta\mu}g_{\nu\rho})\end{aligned} \quad (1)$$

Note: You do not have to derive this or any other identity, but only give a prescription to derive such an identity. **Hint: An iterative approach works best.**

11-2, 35 pts. *Feynman parameters.* When we study loop integrals, we will necessarily have to combine multiple propagators with different demonators. The standard tool is the introduction of Feynman parameters (read pp. 189-190 of Peskin and Schroeder). In preparation, we will prove the necessary mathematical identities.

A, 10 pts. Prove by explicit integration that

$$\int_0^1 dx\ dy\ \delta(x+y-1)\frac{1}{(xA+yB)^2} = \frac{1}{AB}\ . \quad (2)$$

B, 10 pts. Prove by explicit differentiation (of the above w.r.t B) that

$$\frac{1}{AB^n} = \int_0^1 dx\ dy\ \delta(x+y-1)\frac{n\ y^{n-1}}{(xA+yB)^{n+1}}\ . \quad (3)$$

C, 15 pts. Prove by induction that

$$\frac{1}{A_1 A_2 \dots A_n} = \int_0^1 dx_1 \dots dx_n \; \delta(\sum x_i - 1) \frac{(n-1)!}{(x_1 A_1 + x_2 A_2 + \dots x_n A_n)^n} \; . \tag{4}$$

11-3, 20 pts. *QED, soft photons, and bremsstrahlung* After reading chapter 6, section 6.1 from Peskin and Schroeder, write a paragraph explaining how the infrared singularity in QED arises from emission of soft photons.

11-4, 30 pts. *Ward-Takahashi identity* Read the discussion about the Ward-Takahashi identity in Section 5.5 of Peskin and Schroeder (specifically p. 160), as well as the fuller (and somewhat overly formal) discussion in Section 7.4. For (A) and (B), you can assume the Compton scattering matrix element from the end of Lecture 22:

$$i\mathcal{M}_{\text{tot}} = -ie^2 \epsilon^*_\nu(p_3) \epsilon_\mu(p_2) \left[\bar{u}(p_4) \left(\frac{\gamma^\nu \not{p}_2 \gamma^\mu + 2p_1^\mu \gamma^\nu}{2p_1 \cdot p_2} + \frac{-\gamma^\mu \not{p}_3 \gamma^\nu + 2p_1^\nu \gamma^\mu}{-2p_1 \cdot p_3} \right) u(p_1) \right] \; . \tag{5}$$

Alternately, you can use the matrix element in equation 5.74 of Peskin and Schroeder, but note the different labeling of the momenta and the upward time flow of the diagrams.

A, 10 pts. Verify the Ward-Takahashi identity for the photon on the μ vertex. Namely, replace $\epsilon_\mu(p_2)$ in the matrix element by $p_{2\mu}$ and show that the matrix element vanishes.

B, 10 pts. Verify the Ward-Takahashi identity for the photon on the ν vertex.

C, 10 pts. Write a paragraph explaining the meaning of the Ward-Takahashi identity and how it is connected to gauge invariance.