Lecture 6.
Quantum Field Theory I.
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Last time: Complex scalar fields, causality, propagator.
Today: Fermions

With the Feynman prescription + free scalar field theory ~~cov~~ covered, we now have a choice.

(i) Consider an interacting scalar field theory.
This will necessarily be a theory with ϕ^3 or ϕ^4 terms in the Lagrangian: $\mathcal{L} \supset \mathcal{L}_{\text{free}} - a\phi^3 - b\phi^4 \ldots$

(ii) Switch to free theory of fermionic fields.
This will be the same exercise ~~we~~ have already done for scalars, but the effects of spin are non-trivial.

We will cover both topics in this course, but there are numerous complications that arise in an <u>interacting</u> field theory that are not present for <u>free</u> field theories. We will also see that more interesting field theories + interactions are possible when the free scalars + free fermions have already been presented, so ~~in~~ in this course, we will take topic (ii) first.

Essentially, from lecture 2, we know that in- and out-states of the S-matrix fall into ~~spin~~ half-integer representations of the Lorentz group, which we call spin.

Recall that the algebra of Lorentz symmetry was constructed of 3 rotations and 3 boosts, which we denoted by J_i and K_i. We then reshuffled these generators into J_i^+ and J_i^-, which demonstrated that the group $SO(1,3)$ was isomorphic to $SU(2) \times SU(2)$.

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k \\
[J_i, K_j] &= i\epsilon_{ijk} K_k \\
[K_i, K_j] &= -i\epsilon_{ijk} J_k
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
J_i^+ &\equiv \tfrac{1}{2}(J_i + iK_i) \\
J_i^- &\equiv \tfrac{1}{2}(J_i - iK_i) \\
[J_i^+, J_j^+] &= i\epsilon_{ijk} J_k^+ \\
[J_i^-, J_j^-] &= i\epsilon_{ijk} J_k^- \\
[J_i^+, J_j^-] &= 0
\end{aligned}
$$

What representation should we consider for $SU(2) \times SU(2)$? The most convenient choice is the language of spin $1/2$ states from quantum mechanics, where we can identify each set of J_i^+ and J_i^- with the Pauli matrices:

$\{A,B\}$ = anticommutator, $\{A,B\} = AB + BA$

Recall, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. Note also $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

So, to get the correct normalization, the generators of any $SU(2)$ group are called $\tau_i \equiv \frac{\sigma_i}{2}$, and they act on 2-component vectors. Note also that τ_i are Hermitian.

Since each $SU(2)$ group is separate, there are two fundamental types of 2-component spinors from Lorentz symmetry. We distinguish them by <u>handedness</u>, calling them left-hand or right-handed.

Assigning the fundamental representation to the number $\frac{1}{2}$ under $SU(2)$, the Left-handed 2-component spinor transforms as $(\frac{1}{2},0)$ under $SU(2)\times SU(2)$, while the right-handed spinor transforms as $(0,\frac{1}{2})$. These are Weyl-spinors (referring to the fact that they are only 2-components), and they are the irreducible representations of the Lorentz group. Given θ_j rotation angles and β_j boost angles, we have a Lorentz transformation of each spinor as

$$\psi_L \to e^{\frac{1}{2}(i\theta_j\sigma_j - \beta_j\sigma_j)}\psi_L = (1+\tfrac{i}{2}\theta_j\sigma_j - \tfrac{1}{2}\beta_j\sigma_j + \dots)\psi_L$$

$$\psi_R \to e^{\frac{1}{2}(i\theta_j\sigma_j + \beta_j\sigma_j)}\psi_R = (1+\tfrac{i}{2}\theta_j\sigma_j + \tfrac{1}{2}\beta_j\sigma_j + \dots)\psi_R$$

Aside: the $(\frac{1}{2},0)$ rep. has $\vec{J}^- = \frac{1}{2}\vec{\sigma}$, $\vec{J}^+ = 0$,

so $\vec{J} = \frac{1}{2}\vec{\sigma}$ + $\vec{K} = \frac{i\vec{\sigma}}{2}$,

while the $(0,\frac{1}{2})$ rep. has $\vec{J}^- = 0$, $J^+ = \frac{1}{2}\vec{\sigma}$,

so $\vec{J} = \frac{1}{2}\vec{\sigma}$ + $\vec{K} = -\frac{i}{2}\vec{\sigma}$,

which gives rise to the sign difference of the boost angles.

Note also that spinors are complex representations,

$$\delta\psi_L = \tfrac{1}{2}(i\theta_j - \beta_j)\sigma_j\psi_L$$

$$\delta\psi_L^\dagger = \tfrac{1}{2}(-i\theta_j - \beta_j)\psi_L^\dagger\sigma_j$$

$$\delta\psi_R = \tfrac{1}{2}(+i\theta_j + \beta_j)\sigma_j\psi_R$$

$$\delta\psi_R^\dagger = \tfrac{1}{2}(-i\theta_j + \beta_j)\psi_R^\dagger\sigma_j$$

(sign difference on θ to P&S vs. Schwartz, here follows Schwartz)

We now construct spinor fields, which are simply spinor-valued functions of spacetime. We can have two types:

2-comp. LH Weyl spinor : $\psi_L(x) = \begin{pmatrix} \chi_1(x) \\ \chi_2(x) \end{pmatrix}$

2-comp. RH Weyl spinor : $\psi_R(x) = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \end{pmatrix}$

Then, using 2-component language, Lorentz transformations are implemented as above.

It is convenient (and more standard) to consider 4-component fermions, known as Dirac fermions.

To introduce them, we consider again the Lorentz algebra. In differential form, we can express the generators of rotations and boosts using

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu).$$

We again calculate the commutator: (see HW3)

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})$$

Any object that obeys this same commutation relation will also implement Lorentz transformations.

To that end, we now construct the Dirac matrices (in the Weyl / chiral basis).

Suppose there exists four $n \times n$ matrices γ^μ that gave the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \cdot \mathbb{1}_{n\times n}$$

Then, an n-dimensional representation of the Lorentz algebra can be constructed as $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

Exercise: Show $J^{\mu\nu} = S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfies the commutation rule of Lorentz algebra.

$$[S^{\mu\nu}, S^{\rho\sigma}] = S^{\mu\nu}S^{\rho\sigma} - S^{\rho\sigma}S^{\mu\nu}$$

$$= \frac{-1}{16}\Big((\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho) - (\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\Big)$$

Drop γ: label by Lorentz indices

$$= \frac{-1}{16}\Big(\mu\nu\rho\sigma - \nu\mu\rho\sigma - \mu\nu\sigma\rho + \nu\mu\sigma\rho$$
$$- \rho\sigma\mu\nu + \sigma\rho\mu\nu + \rho\sigma\nu\mu - \sigma\rho\nu\mu \Big)$$

Use $\gamma^\alpha\gamma^\beta = 2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha$: we can then move out $g^{\alpha\beta}$ since it is identity in $n\times n$ space. Also, $g^{\alpha\beta} = g^{\beta\alpha}$.

$$= \frac{-1}{16}\Big(2g^{\nu\rho}\,\mu\sigma - \mu\rho\nu\sigma - 2g^{\mu\rho}\,\nu\sigma + \nu\rho\mu\sigma$$
$$- 2g^{\nu\sigma}\,\mu\rho + \mu\sigma\nu\rho + 2g^{\mu\sigma}\,\nu\rho - \nu\sigma\mu\rho$$
$$- 2g^{\mu\sigma}\,\rho\nu + \rho\mu\sigma\nu + 2g^{\mu\rho}\,\sigma\nu - \sigma\mu\rho\nu$$
$$+ 2g^{\nu\sigma}\,\rho\mu - \rho\nu\sigma\mu - 2g^{\nu\rho}\,\sigma\mu + \sigma\nu\rho\mu \Big)$$

$$= \frac{-1}{16}\Big(2g^{\nu\rho}[\mu,\sigma] - 2g^{\mu\rho}[\nu,\sigma] - 2g^{\nu\sigma}[\mu,\rho] + 2g^{\mu\sigma}[\nu,\rho]$$
$$- \mu\rho\nu\sigma + \nu\rho\mu\sigma + \mu\sigma\nu\rho - \nu\sigma\mu\rho$$
$$+ \rho\mu\sigma\nu - \sigma\mu\rho\nu - \rho\nu\sigma\mu + \sigma\nu\rho\mu \Big)$$

$$= \frac{-1}{16}\Big(\frac{8}{i} g^{\nu\rho} S^{\mu\sigma} - \frac{8}{i} g^{\mu\rho} S^{\nu\sigma} - \frac{8}{i} g^{\nu\sigma} S^{\mu\rho} + \frac{8}{i} g^{\mu\sigma} S^{\nu\rho}$$
$$- 2g^{\mu\rho}\,\nu\sigma + \rho\mu\nu\sigma + 2g^{\nu\rho}\,\mu\sigma - \rho\nu\mu\sigma$$
$$+ 2g^{\mu\sigma}\,\nu\rho - \sigma\mu\nu\rho - 2g^{\nu\sigma}\,\mu\rho + \sigma\nu\mu\rho$$
$$+ \rho\mu\sigma\nu - \sigma\mu\rho\nu - \rho\nu\sigma\mu + \sigma\nu\rho\mu \Big)$$

$$= \frac{-1}{16}\Big(\frac{8}{i}\big(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\big)$$
$$- 2g^{\mu\rho}\,\nu\sigma + 2g^{\nu\rho}\,\mu\sigma + 2g^{\mu\sigma}\,\nu\rho - 2g^{\nu\sigma}\,\mu\rho$$
$$+ 2g^{\nu\sigma}\,\rho\mu - 2g^{\sigma\mu}\,\rho\nu - 2g^{\nu\rho}\,\sigma\mu + 2g^{\mu\rho}\,\sigma\nu \Big)$$

$$= \frac{-1}{16}\Big(\frac{8}{i}\big(g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho} + g^{\mu\sigma} S^{\nu\rho} \big)$$

$$- 2 g^{\mu\rho} [\nu,\sigma] + 2 g^{\nu\rho} [\mu,\sigma] + 2 g^{\mu\sigma} [\nu,\rho]$$

$$- 2 g^{\nu\sigma} [\mu,\rho] \Big)$$

$$= i \left(g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho} + g^{\mu\sigma} S^{\nu\rho} \right) \quad \checkmark$$

Next time: Explicit Dirac matrices + 4-component Dirac fermions.