Quantum Field Theory
Lecture 4.
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last time: Scalar fields.
Today: Dynamics of free scalar fields.

Having introduced the definition of the free quantum scalar field last time, we now move to the question of dynamics of scalar fields.

To that end, we should understand conservation laws and their relations to underlying symmetries. Conservation laws will necessarily dictate that some quantity is invariant over time, which we usually denote as a charge. Symmetries play a central role in particle physics, condensed matter physics, and beyond, so we highlight how they constrain + are implemented in QFT.

The central result ~~con~~ connecting symmetries + conservation laws is Noether's theorem (after Emmy Noether, German mathematician), which states that:

Every continuous symmetry has a corresponding current $J_\mu(x)$ that is conserved.

We characterize continuous symmetries by performing transformations of fields that leave the action invariant. "Continuous" imposes the restriction that the transformation must be smoothly deformable from the identity via an infinitesimal parameter ϵ: take a transformation as some fcn. of fields

$$\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \epsilon F_i(\phi_j(x), \partial_\mu \phi_j(x)),$$

then $S[\phi_i'] = S[\phi_i]$

action using transformed fields — original action w/ untransformed fields.

Examples include translations (shift $x \to x+\epsilon$), Lorentz rotations and boosts, or phase rotations.

For a nontrivial example, we can consider a complex scalar field, which can be thought of as a scalar field with each spacetime point being assigned a complex number, or as two separate real fields that give two real numbers (with a phase offset like $z = a+bi$).

The Lagrangian is now

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2,$$

where the numerical coefficients are 1 instead of $\frac{1}{2}$ to account for the fact that there are 2 real degrees of freedom.

We can perform a phase rotation $\phi \to e^{i\alpha}\phi$ for arbitrary phase α. Then $\phi^* \to e^{-i\alpha}\phi^*$. It is straightforward to see that (given α is a constant and has no spacetime dependence) the Lagrangian is invariant under this transformation. We can construct the conserved current as $j^\mu = i[(\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)]$

or, more generally, for $\mathcal{L} \to \mathcal{L} + \epsilon \partial_\mu \mathcal{J}^\mu$,

$$\star \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} F_i - \mathcal{J}^\mu .$$

For the complex scalar case, $\mathcal{J}^\mu = 0$ since $\mathcal{L}$ itself is invariant and

$$j^\mu \propto (\partial^\mu \phi^*) \phi - (\partial^\mu \phi) \phi^* ,$$ as desired.

This will be central to our discussion of scalar QED.

Given j^μ from $\star$ above, the conserved charge is $Q = \int d^3x \, j^0$ and the conservation law is $\partial_\mu j^\mu = 0$.

[Aside: complex scalar field needs separate ladder operators for + and - momentum modes. Will see details later.]

Dynamics of fields.

The simplest object to study dynamics of fields is the amplitude for a particle to propagate from y to x, which we can evaluate as $\langle 0 | \phi(x) \phi(y) | 0 \rangle$.

Explicitly, we only get a non-vanishing result when we have $a_{\vec{p}} a^\dagger_{\vec{q}}$ inside since everything else annihilates the vacuum or has the wrong occupancy number.

Since $\langle 0 | a_{\vec{p}} a^\dagger_{\vec{q}} | 0 \rangle = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$,

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \equiv D(x-y)$$

Note this is Lorentz invariant.

Is this amplitude causal? Have to check separately for timelike and spacelike separations.

A necessary condition for satisfying causality is that commutators for spacelike intervals vanish.

Logic: If commutator vanishes, can measure two operators independently.

Calculate $[\phi(x), \phi(y)]$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right)$$

$$\equiv D(x-y) - D(y-x)$$

Calculate explicitly for time like separation. Since Lorentz invariant, choose frame with $(x-y)^2 = t^2 > 0$, $x^0 - y^0 = t$, $\vec{x} = \vec{y}$.

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

$d^3p = d|\vec{p}|\,|\vec{p}|^2\, d\Omega$ as usual.

$$= \int_0^\infty \frac{dp}{(2\pi)^3} \cdot \int d\Omega \cdot \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\cdot t}$$

$$= \frac{1}{4\pi^2} \int_0^\infty dp \frac{p^2}{\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\, t}$$

$$E = \sqrt{p^2+m^2}$$

$$dE = \frac{1}{2\sqrt{p^2+m^2}} \cdot 2p\, dp \Rightarrow E\, dE = p\, dp$$

$p=0 \Rightarrow E=m$

$$= \frac{1}{4\pi^2} \int_m^\infty dE \cdot \frac{E}{p} \cdot \frac{p^2}{E} e^{-iEt}$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2}\, e^{-iEt}$$

For $t \to \infty$, e^{-iEt} oscillates rapidly & the integral vanishes unless we get a slower oscillation. The slowest occurs when we minimize E, which happens for the lower limit of integration, $E=m$.

$$D(x-y) \sim e^{-imt} \quad \text{for timelike sep., } t \to \infty.$$

For spacelike separation, choose frame with

$$(x-y)^2 = -r^2, \quad x^0 - y^0 = 0, \quad \vec{x} - \vec{y} = \vec{r}$$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_{\vec{p}}} \int d\cos\theta \; e^{ipr\cos\theta}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2}{2E_{\vec{p}}} \left. \frac{e^{ipr\cos\theta}}{ipr} \right|_{\cos\theta=-1}^{+1}$$

WLOG, set $\vec{p}$ along $\hat{z}$. Then $\cos\Delta\phi$ starts at π and goes to 0.

$$= \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2}{2E_{\vec{p}}} \cdot \frac{e^{+ipr} - e^{-ipr}}{ipr}$$

$$= \frac{(-i)}{2r(2\pi)^2} \int_{-\infty}^{\infty} dp \frac{p}{\sqrt{p^2+m^2}} e^{+ipr}$$

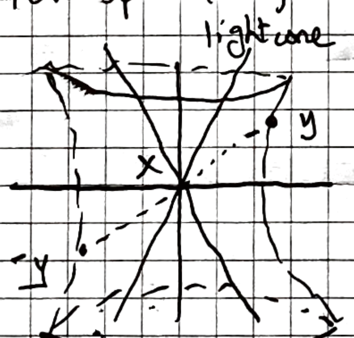
Poles for complex $p^2 = -m^2$, $p = \pm im$.

Can use residue to get

$$D(x-y) = \frac{1}{4\pi^2 r} \int_m^\infty dp \frac{p e^{-pr}}{\sqrt{p^2-m^2}} \to e^{-mr}$$

dominated by $p \to m$

For spacelike, we can smoothly transform $(x-y) \to -(x-y)$.



So, $D(x-y) - D(y-x)$ cancels.

In fact, complex scalar field has important illustration of causality preservation. (Makes particles + antiparticles distinct.)

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a_{\vec{p}} e^{-ip\cdot x} + b_{\vec{p}}^\dagger e^{ip\cdot x} \right]$$

Note on $e^{\mp ip\cdot x}$ + time dependence.

In Heisenberg picture,

$$\phi(x) = \phi(\vec{x},t) = e^{iHt}\,\phi(\vec{x})\,e^{-iHt}$$

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_p e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{-i\vec{p}\cdot\vec{x}}\right)$$

We know that

$$[H, a_{\vec{p}}^\dagger] = E_{\vec{p}}\, a_{\vec{p}}^\dagger \quad \text{and}$$

$$[H, a_{\vec{p}}] = -E_{\vec{p}}\, a_{\vec{p}},$$

so $$H a_{\vec{p}} = a_{\vec{p}}(H - E_{\vec{p}})$$

Consequently, $$H^n a_{\vec{p}} = a_{\vec{p}}(H - E_{\vec{p}})^n$$

and $$H a_{\vec{p}}^\dagger = a_{\vec{p}}^\dagger(H + E_{\vec{p}}), \quad H^n a_{\vec{p}}^\dagger = a_{\vec{p}}^\dagger(H - E_{\vec{p}})^n$$

So, $\cancel{e^{iHt}}$

$$e^{iHt} a_{\vec{p}} = a_{\vec{p}}\, e^{i(H-E_{\vec{p}})t}$$

$$\Rightarrow e^{iHt} a_{\vec{p}}\, e^{-iHt} = a_{\vec{p}}\, e^{-iE_{\vec{p}}t}$$

and also $$e^{iHt} a_{\vec{p}}^\dagger e^{-iHt} = a_{\vec{p}}^\dagger e^{iE_{\vec{p}}t}.$$

We can now get

$$\phi(x) = \phi(\vec{x},t) =$$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\Big(e^{iHt} a_{\vec{p}}\, e^{-iHt} e^{i\vec{p}\cdot\vec{x}} + e^{iHt} a_{\vec{p}}^\dagger e^{-iHt} e^{-i\vec{p}\cdot\vec{x}}\Big)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\, e^{-iE_{\vec{p}}t} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger e^{iE_{\vec{p}}t} e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\, e^{-ip\cdot x} + a_{\vec{p}}^\dagger e^{ip\cdot x}\right)\Big|_{p^0 = E_{\vec{p}}}$$