Quantum Field Theory 1.

Lecture 3.

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Last time: Poincaré symmetry + the irreps of Lorentz symmetry and spin.

Today: Scalar fields

Having established that Poincaré invariance on the S-matrix gives us the simplifying characterization of spin to implement Lorentz invariance / covariance for external states, we study fields with specific spin assignments. Given the $O(1,3) = SU(2) \otimes SU(2)$ symmetry group of Lorentz symmetry, we have half-integer total spin assignments for fields:

$$\phi(x) = \text{scalar, spin} = 0$$
$$\Psi_\alpha(x) = \text{fermion spin} = 1/2$$
$$A_\mu(x) = \text{vector, spin} = 1$$
$$[\; g_{\mu\nu}(x) = \text{graviton, spin} = 2 \;]$$

(Technical complication: The finite dimensional representations from the Lorentz group generated by $SU(2) \times SU(2)$ are not unitary; need $\Lambda^\dagger \Lambda = 1$ s.t. $\langle \Psi | \Psi \rangle = \langle \Psi | \Lambda^\dagger \Lambda | \Psi \rangle$, but since $\Lambda = e^{i\lambda}$, need Hermitian generators. While rotations are Hermitian generators, the boosts, as implemented by $\vec{J}_\pm = \frac{1}{2}(\vec{J} \pm i\vec{K})$, are anti-Hermitian + hence the transformations are anti-unitary. To solve this, we make an infinite-dimensional representation by making the basis depend on p_μ [need to distinguish little-group representations for massive particles ~ $p^\mu = (m, 0, 0, 0)$ or massless particles ~ $p^\mu = (E, 0, 0, E)$, with massive as unitary reps. of $SO(3)$ and massive as unitary reps. of $ISO(2)$. See Schwartz, Sect. 10.2.1.)

(Technical note #2: For elementary particles in 4D spacetime, we can prove an inconsistency between a non-trivial S-matrix and long-range forces mediated by particles with spin > 2. We can also establish the uniqueness of the graviton as the only massless spin-2 field + the fact that the only long-range force it mediates is gravity. So, we are already very limited in the choices of possible spins for elementary fields!
Refs: Benincasa, Cachazo, [0705.4305];
Weinberg, Phys. Rev. 134 (1964) B882,
Phys. Rev. 135 (1964) B1049,
Phys. Rev. 138 (1965) B988.
Proof (heuristic) based on amplitudes + consistent factorization of S-matrix: Baumann, What Long-range forces are allowed?)

Start with scalar fields. Recall that fields are number- (or object-) valued functions defined for all spacetime points. Scalar fields are the simplest, since $\phi(x)$ just gives a pure number for a given $x = (t, x, y, z)$. Also familiar should be electric fields from classical field theory, where each point gives or magnetic a three-vector $\vec{E}(x)$ or $\vec{B}(x)$.

As classical fields, it is straightforward to build arbitrary field configurations by superposition. Recall that $\vec{E} = -\vec{\nabla}\Phi$, for $-\nabla^2 \Phi = \rho(x)$, so we could ~~~~ always split up the ~~cha~~ source charge distribution, solve for the potential + $\vec{E}$ field from each piece separately, then superpose the individual $\Phi(x)$ + $\vec{E}(x)$ solutions to get the total.

In quantum theory, this is no longer true generally, if the solutions belong to different eigenbases. While distinct position-space solutions to Schrödinger's eqn ~~are~~ obey superposition, we cannot solve for separate position + momentum space solutions and superpose them. The key, as usual, is the commutation of operators allows superposition while non-commutation does not:

$$[\hat{x},\hat{p}] = i\hbar \quad , \quad [\hat{x},\hat{x}] = [\hat{p},\hat{p}] = 0$$

By analogy, in quantum theories, we can never know precisely a given state and its conjugate momentum simultaneously. Recall from classical Hamiltonian theory that the principle of least action governs the trajectory of a given degree of freedom, with $S = \int L\,dt = \int \mathcal{L}(\phi, \partial_\mu\phi)\, d^4x$ $\quad \left[\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\right]$

Here, $\mathcal{L}$ is the Lagrangian density (dimensions of $(\text{length})^{-4}$ $= (\text{energy})^4$), but we will always refer to it as the Lagrangian.

We find the extrema of S by performing a variation w.r.t. ϕ and $\partial_\mu\phi$:

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right\}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) - \left(\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right)\delta\phi\right\}$$

The second term is the integral of a total derivative, so we only need to evaluate it at the bdry of spacetime.

Assuming fields vanish at bdry, we get

$$0 = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi ,$$

and since $\delta\phi$ is arbitrary, the integrand vanishes:

Euler-Lagrange equations of motion

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$; straightforward to extend $\phi \to \phi_i$.

In Hamiltonian theory, (closer analogy to QM + more obvious/explicit implementation quantization), we construct

$$S = \int dt \; L(q_i, \dot{q}_i) \Rightarrow H(q_i, p_i) = \sum_i p_i \dot{q}_i - L$$

with $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$.

We will match coordinates in classical mechanics with <u>generalized</u> <u>coordinates</u> (for scalar field theory) that are labeled by given $(t, \vec{x}) \Rightarrow \phi(t, \vec{x}_n) \equiv \phi_n(t)$ over a discretized lattice spacing.

Then conjugate momentum to $\phi(\vec{x})$ is

$$p(\vec{x}) \equiv \frac{\partial L}{\partial \dot{\phi}(\vec{x})} = \frac{\partial}{\partial \dot{\phi}(\vec{x})} \int d^3y \; \mathcal{L}(\phi(\vec{y}), \dot{\phi}(\vec{y}))$$

discretization, replace integral by sum

$$\sim \frac{\partial}{\partial \dot{\phi}(\vec{x})} \sum_{\vec{y}_n} \Delta V \, \mathcal{L}[\phi(\vec{y}_n), \vec{\nabla}\phi(\vec{y}_n), \dot{\phi}(\vec{y}_n)]$$

L to $\int \mathcal{L} d^3y$

finite volume element

$$= \pi(\vec{x}) \, \Delta V$$

for $\pi(\vec{x}) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x})}$ as the momentum density conjugate to $\phi(\vec{x})$.

Restoring the continuum,

$$H = \sum_{\vec{x}_n} p(\vec{x}_n) \, \dot{\phi}(\vec{x}_n) - L$$

$$= \sum_{\vec{x}_n} \pi(\vec{x}_n) \, \Delta V \, \dot{\phi}(\vec{x}_n) - L$$

$$\simeq \int d^3x \, [\pi(\vec{x}) \dot{\phi}(\vec{x}) - \mathcal{L}] \equiv \int d^3x \, \mathcal{H}$$

Hamiltonian density

Side note: with Lagrange formalism, easier to see Lorentz invariance

Thus far, this has entirely been classical field theory. To obtain quantum field theory, we need to promote coordinates to operators and impose commutation relations:

position + momentum coords. as operators:
$$[x_i, p_i] = i\delta_{ij}$$
$$[x_i, x_j] = [p_i, p_j] = 0$$

For $\phi_i(t, \vec{x})$ and $\pi_j(t, \vec{y})$ in Hamiltonian formalism, we impose equal-time commutation relations:

coordinate + momentum densities
$$[\phi_i(t,\vec{x}), \pi_j(t,\vec{y})] = i\delta_{ij}\, \delta^{(3)}(\vec{x}-\vec{y})$$
$$[\phi_i(t,\vec{x}), \phi_j(t,\vec{y})] = [\pi_i(t,\vec{x}), \pi_j(t,\vec{y})] = 0$$

Use Heisenberg picture, where operators ϕ and π are time-dependent, and Hamiltonian also carries time-dependence and is treated as an operator. We have the Heisenberg EOM,

$$i\frac{\partial}{\partial t} O = [O, H]$$

$$\Rightarrow \quad \partial_t \phi_j(t,\vec{x}) = \frac{1}{i}[\phi_j(t,\vec{x}), H]$$

Ex: check this: $\Rightarrow \quad \phi_j(t,\vec{x}) = e^{iHt}\, \phi_j(0,\vec{x})\, e^{-iHt}$

Note that imposing commutation relations at $t = t_0$ preserves them for all time t:

$$\begin{aligned}[\phi_i(t,\vec{x}), \pi_j(t,\vec{y})] &= [e^{iH(t-t_0)}\phi_i(t_0,\vec{x})e^{-iH(t-t_0)}, e^{iH(t-t_0)}\pi_j(t_0,\vec{y})e^{-iH(t-t_0)}] \\ &= e^{iH(t-t_0)}[\phi_i(t_0,\vec{x}), \pi_j(t_0,\vec{y})]e^{-iH(t-t_0)} \\ &= i\delta_{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\end{aligned}$$

For arbitrary H, ~~f~~ or equivalently, arbitrary potentials, there exact quantum system is not solvable.

Recall EOM : real scalar

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$$

$$V(\phi) = V_0 + \frac{m^2}{2} \phi^2 + \frac{\lambda_3}{3!} \phi^3 + \frac{\lambda_4}{4!} \phi^4$$

(linear ~~shift~~ term removed by shift)

Euler - Lagrange EOM:

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{-\partial V}{\partial \phi} \quad , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\underbrace{\partial_\mu \partial^\mu}_{\Box} \phi + \frac{\partial V}{\partial \phi} = 0$$

$$\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \equiv \Box = \text{d'Alembertian}$$

$$(\Box + m^2)\phi = \underbrace{-\frac{\lambda_3}{2} \phi^2 - \frac{\lambda_4}{3!} \phi^3 + \ldots}$$

non-linearities = self-interactions
= inhomogeneous wave eqn.

If we turn off all powers ϕ^3 & higher in $V(\phi)$, we get the familiar <u>Klein-Gordon eqn.</u>

$$(\Box + m^2)\, \phi(x) = 0$$

We solve by the usual ~~[illegible]~~ Fourier transformation,

$$\phi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

with associated

$$\pi(\vec{x}) = \int \frac{d^3 p}{(2\pi)} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

Justification: Fourier transform of free KG eqn. lets us solve the quantum system via ladder operators from the simple harmonic oscillator associated with momentum $|\vec{p}|$ and mass m:

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega_p^2 \phi^2, \qquad \omega_p = \sqrt{|\vec{p}|^2 + m^2}$$

Review: construct eigenstates as occupation numbers of eigenmodes,

$$\phi = \frac{1}{\sqrt{2\omega}} (a + a^\dagger), \qquad p = -i\sqrt{\frac{\omega}{2}} (a + a^\dagger).$$

Then $[\phi, p] = i \Rightarrow [a, a^\dagger] = 1$,

so $H = \omega\left(a^\dagger a + \frac{1}{2}\right)$.

We build the spectrum by setting $a|0\rangle = 0$ with eigenvalue $\frac{1}{2}\omega$. We also have $[H, a^\dagger] = \omega a^\dagger$, $[H, a] = -\omega a$,

so $|n\rangle = (a^\dagger)^n |0\rangle$ are eigenstates with eigenvalues $(n + \frac{1}{2})\omega$.

In other words, $\phi(x)$ is (for free scalar theory!) a superposition of momentum wavepackets indexed by $\vec{p}$, and $\phi(\vec{x})$ operator creates a particle at position $\vec{x}$.

continuous sum over $\vec{p} \Rightarrow$ integral over momentum space

We can also find the equivalent commutation relation for ladder operators,

$$[a(\vec{p}), a^\dagger(\vec{p}')] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}'),$$

with others vanishing.

$a(\vec{p})$ = annihilation, $a^\dagger(\vec{p})$ = creation.

Define $|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}} |0\rangle$. (momentum wavepacket)

Then $\phi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle$ and

$\langle 0 | \phi(\vec{x}) | \vec{p}\rangle = e^{i\vec{p}\cdot\vec{x}}$.