Vacuum Polarization & Wavefunction Renormalization of Photons and Fermions in QED.

Quantum Field Theory 1.

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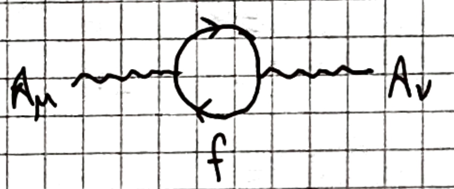
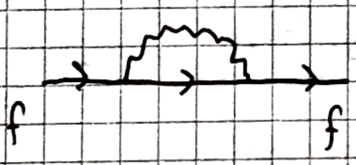
In this note, we perform the 1-loop calculation for external states in QED. We have two diagrams to calculate:

1. "Wavefunction renormalization" / "Vacuum polarization" diagram for photons.



2. "Wavefunction renormalization" for fermions



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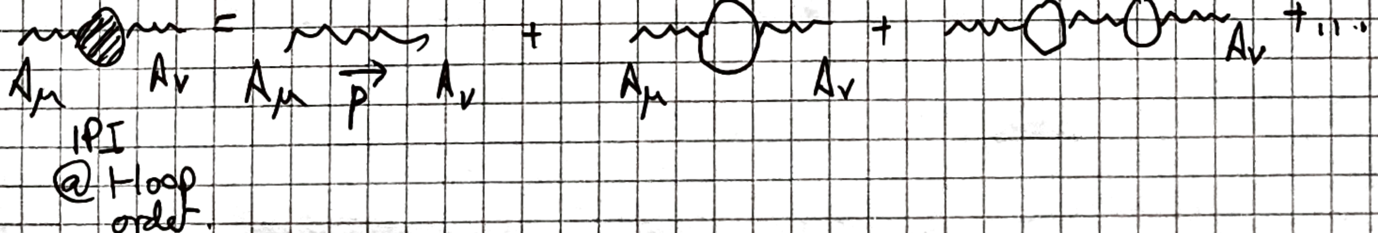
1. We start with the vacuum polarization diagram.

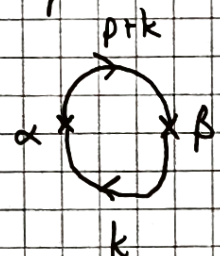
In general, we will always consider a resummation of one-loop corrections to an external momentum eigenstate. Consequently, we have to consider the following series of diagrams:



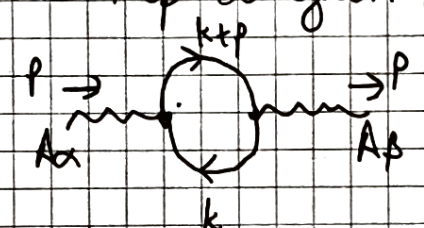
The crucial element to calculate is thus the truncated matrix element,

, where the (α, β) Lorentz indices are then matched to either external photon states directly or possibly to propagating photons. This truncated diagram is known as the vacuum polarization (at 1-loop), and is denoted by

$i\Pi_{\alpha\beta} =$



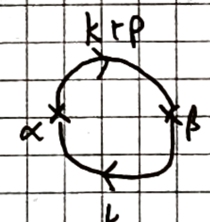
, where, by definition, the full one-loop diagram is



$= i\mathcal{M} \equiv i\epsilon_\alpha(p)\epsilon^*_\beta(p)\,\Pi_{\alpha\beta}(p^2)$

Hence, we can always truncate the external photons of the full one-loop diagram to get the vacuum pol., which has two <u>uncontracted</u> Lorentz indices.

$i\Pi_{\alpha\beta}(p^2) =$



closed fermion loop

$$= \underbrace{(-1)}_{\text{fermion loop}} \int \frac{d^4k}{(2\pi)^4} \, \mathrm{Tr}\left[\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2+i\epsilon} \cdot ieQ\gamma^\alpha \cdot \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} \cdot ieQ\gamma^\beta \right]$$

[Note, since this is a closed loop of fermions, we necessarily sum over all states, including possible colors + flavors.]

$$= -e^2Q^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}[(\not k + \not p + m)\gamma^\alpha(\not k + m)\gamma^\beta]}{((k+p)^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)}$$

We use Feynman parameters:

$$= -e^2Q^2 \int dx\,dy\,\delta(x+y-1) \int \frac{d^4k}{(2\pi)^4} \cdot \frac{\mathrm{Tr}[(\not k+\not p+m)\gamma^\alpha(\not k+m)\gamma^\beta]}{(k^2+2xp\cdot k + xp^2 - m^2 + i\epsilon)^2}$$

Complete the square in the denominator & shift loop momentum: (integral over y is trivial)

$$\ell \equiv k + xp, \quad \Delta \equiv x^2p^2 - xp^2 + m^2 + i\epsilon$$

$$= -e^2Q^2\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{\mathrm{Tr}[(\not\ell+\not p - x\not p + m)\gamma^\alpha(\not\ell - x\not p + m)\gamma^\beta]}{(\ell^2-\Delta)^2}$$

Simplify numerator & use oddness of ℓ-integration.

$$\mathrm{Tr}[(\not\ell + \not p(1-x) + m)\gamma^\alpha(\not\ell - x\not p + m)\gamma^\beta]$$

$$= \mathrm{Tr}[\not\ell\gamma^\alpha\not\ell\gamma^\beta] + \mathrm{Tr}[\not p(1-x)\gamma^\alpha(-x\not p)\gamma^\beta] + m^2\,\mathrm{Tr}[\gamma^\alpha\gamma^\beta]$$

$$= 4(\ell^\alpha\ell^\beta - \ell^2 g^{\alpha\beta} + \ell^\beta\ell^\alpha) + (1-x)(-x)4(p^\alpha p^\beta - p^2 g^{\alpha\beta} + p^\beta p^\alpha) + 4m^2 g^{\alpha\beta}$$

$$= 4(2\ell^\alpha\ell^\beta - \ell^2 g^{\alpha\beta}) - 4x(1-x)(2p^\alpha p^\beta - p^2 g^{\alpha\beta}) + 4m^2 g^{\alpha\beta}$$

Use dimensional regularization for loop momenta integrals. Note:

$$\int \frac{d^d\ell}{(2\pi)^d} \frac{\ell^\alpha\ell^\beta}{(\ell^2-\Delta+i\epsilon)^n} = \int \frac{d^d\ell}{(2\pi)^d}\, \underset{\text{not } \frac{1}{4}}{\frac{1}{d}}\, \frac{\ell^2 g^{\alpha\beta}}{(\ell^2-\Delta)^n}$$

Thus, $\mathrm{Tr}[\ldots] = 4\left(\frac{2}{d}-1\right)\ell^2 g^{\alpha\beta} - 4x(1-x)\left(2p^\alpha p^\beta - p^2 g^{\alpha\beta}\right) + 4m^2 g^{\alpha\beta}$

$$\Rightarrow i\Pi_{\alpha\beta}(p) = -e^2Q^2\int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \cdot \frac{4\left[\left(\frac{2}{d}-1\right)\ell^2 g^{\alpha\beta} - x(1-x)\left(2p^\alpha p^\beta - p^2 g^{\alpha\beta}\right) + m^2 g^{\alpha\beta}\right]}{(\ell^2-\Delta)^2}$$

(we discard the mass scaling μ^{4-d}).

We have the loop integral identities in dim. reg.

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^n} = \frac{(-1)^{n-1}}{(4\pi)^{d/2}}\, i\,\frac{d}{2}\,\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1}$$

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}}\,\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2}$$

Plug in:

$$i\Pi_{\alpha\beta}(p) = -e^2Q^2\int_0^1 dx\,(4)\left[\left(\frac{2}{d}-1\right)\frac{(-1)}{(4\pi)^{d/2}}\, i\,\frac{d}{2}\cdot\Gamma\left(1-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} g^{\alpha\beta} + \left(-x(1-x)\left(2p^\alpha p^\beta - p^2 g^{\alpha\beta}\right) + m^2 g^{\alpha\beta}\right)\cdot\frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-d/2}\right]$$

Use $\left(\frac{2}{d}-1\right)\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) = \left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) = \Gamma\left(2-\frac{d}{2}\right)$

$$i\Pi_{\alpha\beta}(p) = -4e^2Q^2\int_0^1 dx\left[\frac{-i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\cdot\frac{\Delta g^{\alpha\beta}}{(\Delta)^{2-d/2}} + \left(-x(1-x)\left(2p^\alpha p^\beta - p^2 g^{\alpha\beta}\right) + m^2 g^{\alpha\beta}\right)\frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-d/2}\right]$$

The pole from the ℓ^2 integral is lifted!

Factor out $\frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-d/2}$. Plug in $\Delta = x^2p^2 - xp^2 + m^2 + i\epsilon$

$$i\Pi_{\alpha\beta}(p) = -4ie^2Q^2\frac{1}{(4\pi)^{d/2}}\int dx\,\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-d/2}\left[-\Delta g^{\alpha\beta} - x(1-x)\left(2p^\alpha p^\beta - p^2 g^{\alpha\beta}\right) + m^2 g^{\alpha\beta}\right]$$

$$i\Pi_{\alpha\beta}(p^2) = -4ie^2Q^2 \frac{1}{(4\pi)^{d/2}} \int dx\, \Gamma(2-d/2)\left[\frac{1}{\Delta}\right]^{2-d/2} \left[-2(1-x)x\,(p^\alpha p^\beta - p^2 g^{\alpha\beta})\right]$$

(we drop $i\epsilon$ since it only plays a role in deforming the integration contour for the Wick rotation.)

$$= 8ie^2Q^2 \frac{1}{(4\pi)^{d/2}} \int dx\, \Gamma(2-d/2)\left(\frac{1}{\Delta}\right)^{2-d/2}(x(1-x))\cdot(p^\alpha p^\beta - p^2 g^{\alpha\beta})$$

We now use, $d = 4-2\epsilon$, and then

$$\frac{1}{(4\pi)^{d/2}}\Gamma(2-d/2)\left(\frac{1}{\Delta}\right)^{2-d/2} \to \frac{1}{16\pi^2}\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log 4\pi\right) + \mathcal{O}(\epsilon)$$

discard

Note: different ϵ compared to $i\epsilon$ in propagators.

$$i\Pi_{\alpha\beta}(p) = \frac{ie^2Q^2}{2\pi^2}\int dx\,(x(1-x))(p^\alpha p^\beta - p^2 g^{\alpha\beta})\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta\right)$$

$$\Pi_{\alpha\beta}(p) = -\frac{e^2Q^2}{2\pi^2}(p^2 g^{\alpha\beta} - p^\alpha p^\beta)\int dx\, x(1-x)\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta\right)$$

Remark: This structure of $p^2 g^{\alpha\beta} - p^\alpha p^\beta$ is guaranteed by the Ward identity & Lorentz covariance.

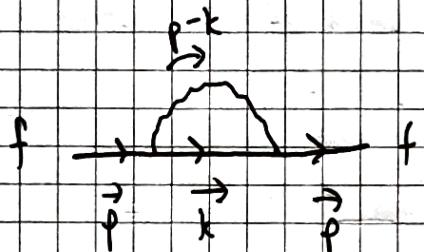
We can define $\Pi_{1\text{-loop}}(p)(p^2 g^{\alpha\beta} - p^\alpha p^\beta) = \Pi^{\alpha\beta}(p)$

so $\Pi_{1\text{-loop}}(p^2)$ is the scalar loop function

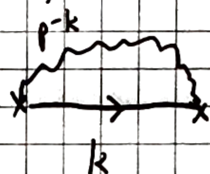
$$\Pi_{1\text{-loop}} = \frac{-e^2Q^2}{2\pi^2}\int dx\, x(1-x)\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta\right)$$

Remark: The divergence as $\epsilon \to 0$ does not vanish & instead requires a <u>counterterm</u>.

We now proceed with the fermion wavefunction:



Again, we treat this as the one-loop correction to the fermion propagator, so we truncate the extra propagator lines and define

$$-i\,\Sigma_2(p) =$$ 

(So the one-loop diagram above would sandwich $-i\Sigma_2(p)$ by $\frac{i(\not{p}+m)}{p^2-m^2}$ on either side.)

For technical reasons related to IR (infrared) divergences, we will find it easiest to calculate the diagram if we include an IR regulator mass for the photon: μ^2. Then, we would have

$$-i\,\Sigma_2(p) = \int \frac{d^4k}{(2\pi)^4} \cdot (ieQ\gamma^\nu) \cdot \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} \cdot (ieQ\gamma^\mu) \cdot \frac{-ig_{\mu\nu}}{(p-k)^2-\mu^2+i\epsilon}$$

$$= -e^2Q^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\mu(\not{k}+m)\gamma^\mu}{(k^2-m^2+i\epsilon)\,((p-k)^2-\mu^2+i\epsilon)}$$

Introduce Feynman parameters.

$$= -e^2Q^2 \int dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\mu(\not{k}+m)\gamma^\mu}{(k^2-2p\cdot kx+xp^2-(1-x)m^2-x\mu^2+i\epsilon)^2}$$

Shift ~~of~~ loop momentum. $l \equiv k - xp$.

$$= -e^2Q^2\int dx \int \frac{d^4l}{(2\pi)^4} \cdot \frac{\gamma_\mu(\not{l} + x\not{p} + m)\gamma^\mu}{(l^2 - x^2p^2 + xp^2 - (1-x)m^2 - x\mu^2 + i\epsilon)^2}$$

Define $\Delta \equiv x^2p^2 - xp^2 + (1-x)m^2 + x\mu^2 - i\epsilon$

We can drop the linear term of $\not{l}$ in the numerator.

Adopt dim. regularization:

$$m\gamma_\mu\gamma^\mu = dm, \qquad \gamma_\mu x(\not{p})\gamma^\mu = -(d-2)x\not{p}$$

$$= -e^2Q^2\int dx \int \frac{d^dl}{(2\pi)^d} \frac{-(d-2)x\not{p} + dm}{(l^2 - \Delta)^2}$$

Again, use the dim. reg. identity.

$$= -e^2Q^2\int dx \frac{i}{(4\pi)^{d/2}} \cdot \frac{\Gamma(2-d/2)}{(\Delta)^{2-d/2}} (-(d-2)x\not{p} + dm)$$

We use the usual expression for $d = 4 - \epsilon$,

$$-i\Sigma_2(p) = -e^2Q^2\int dx \frac{i}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta\right)(-(d-2)x\not{p} + dm)$$

$$= \frac{-ie^2Q^2}{16\pi^2}\int dx \left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta\right)(-(2-\epsilon)x\not{p} + (4-\epsilon)m)$$

$$= \frac{-ie^2Q^2}{16\pi^2}\int dx \left[\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta\right)(-2x\not{p} + 4m) + (2x\not{p} - 2m)\right] \leftarrow \text{extra contribution from dim. reg.}$$

We get
$$\Sigma_2(p) = \frac{e^2Q^2}{16\pi^2}\int dx \left[\left(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta\right)(-2x\not{p} + 4m) + (2x\not{p} - 2m)\right]$$