Lecture 23.
Quantum Field Theory I.
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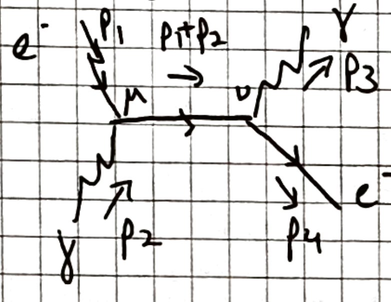
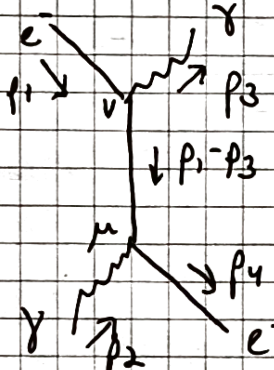
Last time: $e^+e^- \to q\bar{q}$ R ratio, crossing symmetry, Compton scattering
Today: Finish Compton scattering, Klein-Nishina formula

From last time, we discussed Compton scattering: $e^-\gamma$ to $e^-\gamma$

Two diagrams:





We had
$$i\mathcal{M}_{tot} = -ie^2 \epsilon^*_\nu \epsilon_\mu \left[\bar{u}_4 \left(\frac{\gamma^\nu \not{p}_2 \gamma^\mu + 2p_1^\mu \gamma^\nu}{2p_1\cdot p_2} + \frac{-\gamma^\mu \not{p}_3 \gamma^\nu + 2p_1^\nu \gamma^\mu}{-2p_1\cdot p_3} \right) u_1 \right]$$

Now, we square the matrix element: as usual, we will consider unpolarized cross sections, so we need to sum over final states + average over initial states.

$$\sum_{\text{spins, pols.}} |\mathcal{M}|^2 = e^4 \Big(\sum_{\text{pols.}} \epsilon^*_\nu(p_3)\, \epsilon_\beta(p_3)\, \epsilon_\mu(p_2)\, \epsilon^*_\alpha(p_2)\Big)$$
$$\Big\{ \sum_{\text{spins}} \left[\bar{u}_4 \left(\left(\frac{\gamma^\nu \not{p}_2 \gamma^\mu + 2p_1^\mu \gamma^\nu}{2p_1\cdot p_2}\right) + \frac{-\gamma^\mu \not{p}_3 \gamma^\nu + 2p_1^\nu \gamma^\mu}{-2p_1\cdot p_3} \right) u_1 \right]$$
$$\left[\bar{u}_1 \left(\frac{\gamma^\alpha \not{p}_2 \gamma^\beta + 2p_1^\alpha \gamma^\beta}{2p_1\cdot p_2} + \frac{-\gamma^\beta \not{p}_3 \gamma^\alpha + 2p_1^\beta \gamma^\alpha}{-2p_1\cdot p_3} \right) u_4 \right] \Big\}$$

Need to reverse order of γ-matrices in $\mathcal{M}^\dagger$, use α + β dummy indices

Aside: $\left(\bar{u}_4 \gamma^\nu \not{p}_2 \gamma^\mu u_1\right)^\dagger = u_1^\dagger \gamma^{\mu\dagger} (\not{p}_2)^\dagger \gamma^{\nu\dagger} \gamma^0 u_4$

$$= \bar{u}_1 \gamma^\mu \not{p}_2 \gamma^\nu u_4 .$$

We need an identity for the sum over photon polarizations:

The rule is $\sum_{\text{pol.}} \epsilon_\mu(p_2)\, \epsilon^*_\alpha(p_2) = -g_{\mu\alpha}$

when we deal with massless vectors (like the photon).

An imprecise derivation of this rule is given in P&S, p. 160, but a better argument is:

$$\sum_{\lambda = \text{pols. } 1,2} \epsilon^*_\mu(k,\lambda)\, \epsilon_\nu(k,\lambda) = -g_{\mu\nu}$$

is a valid substitution when both sides of the equation are contracted into QED amplitudes satisfying gauge symmetry. In particular, the basis-invariance of a ~~po~~ transverse polarization vector ensures that

$$\sum_{\lambda=1,2} \epsilon^*_\mu(k,\lambda)\, \epsilon_\nu(k,\lambda)\, \mathcal{M}^\mu(k,\ldots)\, \mathcal{M}^{*\nu}(k,\ldots)$$

$$\bigstar \quad = |\mathcal{M}^1(k,\ldots)|^2 + |\mathcal{M}^2(k,\ldots)|^2$$

where we write $\mathcal{M} \equiv \epsilon^*_\mu \mathcal{M}^\mu$ for $\mathcal{M}^\mu$ as the truncated matrix element with the photon vertex uncontracted. Given that QED current is conserved,

$\partial_\mu j^\mu(x) = 0 \Rightarrow k_\mu \mathcal{M}^\mu(k,\ldots) = 0$ (by Fourier transform to momentum space, $\mathcal{M}^\mu$ must correspond to a QED current element carried by photon with momentum k.)

But then a photon $k_\mu = (k,0,0,k)$ (so the transverse pols. are in the xy plane) forces $\mathcal{M}^0(k,\ldots) = \mathcal{M}^3(k,\ldots)$

$$\Rightarrow \bigstar = |\mathcal{M}^1(k,\ldots)|^2 + |\mathcal{M}^2(k,\ldots)|^2 + |\mathcal{M}^3(k,\ldots)|^2 - |\mathcal{M}^0(k,\ldots)|^2 = -g_{\mu\nu}\mathcal{M}^\mu \mathcal{M}^{*\nu} \quad ✓$$

Note here that the longitudinal & time components of M_μ cancelled, as a result of the fact that the time-like photon polarization acts as a negative-norm state. This is generic, and the cancellation will occur in any gauge-invariant QED calculation.

Aside: In fact, a test of whether a group of Feynman diagrams is gauge invariant is testing whether this cancellation applies. (between longitudinal & timelike photon polarizations).

Returning to the calculation, we use

$$\sum_{\text{pols.}} \epsilon^*_\nu(p_3)\,\epsilon_\beta(p_3) = -g_{\nu\beta}, \qquad \sum_{\text{pols.}} \epsilon_\mu(p_2)\,\epsilon^*_\alpha(p_2) = -g_{\mu\alpha}.$$

$$\Rightarrow \sum_{\text{spins}}\sum_{\text{pols}} |M|^2 = e^4 (g_{\nu\beta}\, g_{\mu\alpha}) \times$$

$$\mathrm{Tr}\Big[(\not p_4 + m_e)\Big(\frac{\gamma^\nu \not p_2 \gamma^\mu + 2p_1^\mu \gamma^\nu}{2p_1\cdot p_2} + \frac{-\gamma^\mu \not p_3 \gamma^\nu + 2p_1^\nu \gamma^\mu}{-2p_1\cdot p_3}\Big)$$

$$(\not p_1 + m_e)\Big(\frac{\gamma^\alpha \not p_2 \gamma^\beta + 2p_1^\alpha \gamma^\beta}{2p_1\cdot p_2} + \frac{-\gamma^\beta \not p_3 \gamma^\alpha + 2p_1^\beta \gamma^\alpha}{-2p_1\cdot p_3}\Big)\Big]$$

The main calculational complication is that we have distinct denominators.

We write out the four terms (distinguished by denominators); but note that the traces are much simpler because of the polarization sums giving $g_{\nu\beta}\, g_{\mu\alpha}$.

$$\sum_{\text{spins}}\sum_{\text{pols}} |M|^2 = e^4 \times$$

$$\mathrm{Tr}\Big[(\not p_4 + m_e)\Big(\frac{\gamma^\nu \not p_2 \gamma^\mu + 2p_1^\mu \gamma^\nu}{2p_1\cdot p_2} + \frac{-\gamma^\mu \not p_3 \gamma^\nu + 2p_1^\nu \gamma^\mu}{-2p_1\cdot p_3}\Big)$$

$$(\not p_1 + m_e)\Big(\frac{\gamma_\mu \not p_2 \gamma_\nu + 2p_{1\mu} \gamma_\nu}{2p_1\cdot p_2} + \frac{-\gamma^\nu \not p_3 \gamma^\mu + 2p_1^\nu \gamma^\mu}{-2p_1\cdot p_3}\Big)\Big]$$

4 Traces:

Recall identities:

$$\not p \not p = p^2$$
$$\gamma^\alpha \gamma_\alpha = 4$$
$$\gamma^\mu \gamma^\alpha \gamma_\mu = -2\gamma^\alpha$$
$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma_\mu = 4 g^{\alpha\beta}$$
$$\gamma^\mu \gamma^\alpha \gamma^\beta \gamma^\delta \gamma_\mu = -2 \gamma^\delta \gamma^\beta \gamma^\alpha$$

We also have on-shell momentum invariants,

$$p_1^2 = m_e^2,\quad p_4^2 = m_e^2,\quad p_2^2 = 0,\quad p_3^2 = 0$$
$$s \equiv (p_1+p_2)^2 = m_e^2 + 2p_1\cdot p_2$$
$$t \equiv (p_1-p_3)^2 = (p_4-p_2)^2 = m_e^2 - 2p_1\cdot p_3 = m_e^2 - 2p_2\cdot p_4$$
$$u \equiv (p_1-p_4)^2 = 2m_e^2 - 2p_1\cdot p_4 = (p_3-p_2)^2 = -2p_2\cdot p_3$$
$$s+t+u = 2m_e^2.$$

First trace:

$$\frac{1}{4(p_1\cdot p_2)^2} \mathrm{Tr}\left[(\not p_4 + m_e)(\gamma^\nu \not p_2 \gamma_\mu + 2p_1^\mu \gamma^\nu)(\not p_1 + m_e)(\gamma_\mu \not p_2 \gamma_\nu + 2p_{1\mu}\gamma_\nu)\right]$$

Discard odd # of γ-matrices:

$$= \frac{1}{4(p_1\cdot p_2)^2} \mathrm{Tr}\Big[\not p_4 (\gamma^\nu \not p_2 \gamma^\mu + 2p_1^\mu \gamma^\nu)(\not p_1)(\gamma_\mu \not p_2 \gamma_\nu + 2p_{1\mu}\gamma_\nu) + m_e^2 (\gamma^\nu \not p_2 \gamma_\mu + 2p_1^\mu \gamma^\nu)(\gamma_\mu \not p_2 \gamma_\nu + 2p_{1\mu}\gamma_\nu)\Big]$$

$$= \frac{1}{4(p_1\cdot p_2)^2} \mathrm{Tr}\Big[\not p_4 \gamma^\nu \not p_2 \gamma^\mu \not p_1 \gamma_\mu \not p_2 \gamma_\nu + \not p_4\, 2p_1^\mu \gamma^\nu \not p_1 \gamma_\mu \not p_2 \gamma_\nu + \not p_4 \gamma^\nu \not p_2 \gamma^\mu \not p_1 2p_{1\mu} \gamma_\nu + \not p_4\, 2p_1^\mu \gamma^\nu \not p_1\, 2p_{1\mu}\gamma_\nu + m_e^2\left(0 + 2\gamma^\nu \not p_1 \not p_2 \gamma_\nu + 2\gamma^\nu \not p_2 \not p_1 \gamma_\nu + 4m_e^2 \cdot 4\right)\Big]$$

$$= \frac{1}{4(p_1\cdot p_2)^2} \mathrm{Tr}\Big[-2\not p_4 \gamma^\nu \not p_2 \not p_1 \not p_2 \gamma_\nu - 4\not p_4 \not p_2 m_e^2 - 4\not p_4 \not p_2 m_e^2 - 8m_e^2 \not p_4 \not p_1 + m_e^2\left(8p_1\cdot p_2 + 8p_1\cdot p_2 + 16m_e^2\right)\Big]$$

$$= \frac{1}{4(p_1 \cdot p_2)^2} \mathrm{Tr}\Big[4 \not{p}_4 \not{p}_2 \not{p}_1 \not{p}_2 - 8 \not{p}_4 \not{p}_2 m_e^2 - 8 m_e^2 \not{p}_4 \not{p}_1 + 16 m_e^2 p_1 p_2 + 16 m_e^4 \Big]$$

$$= \frac{1}{4(p_1 \cdot p_2)^2} \Big[16 \big(p_2 \cdot p_4 \, p_1 \cdot p_2 - 0 + p_2 \cdot p_4 \, p_1 \cdot p_2 \big) - 32 m_e^2 p_2 \cdot p_4 - 32 m_e^2 p_1 p_4 + 64 m_e^2 p_1 p_2 + 64 m_e^4 \Big]$$

$$= \frac{1}{(p_1 \cdot p_2)^2} \big(8 p_1 p_2 \, p_2 \cdot p_4 - 8 m_e^2 p_2 \cdot p_4 - 8 m_e^2 p_1 p_4 + 16 m_e^2 p_1 p_2 + 16 m_e^4 \big)$$

Second trace:

$$\frac{-1}{4(p_1 \cdot p_2)(p_1 \cdot p_3)} \mathrm{Tr}\Big[(\not{p}_4 + m_e)(-\gamma^\mu \not{p}_3 \gamma^\nu + 2 p_1^\nu \gamma^\mu)(\not{p}_1 + m_e)(\gamma_\mu \not{p}_2 \gamma_\nu + 2 p_{1\mu} \gamma_\nu) \Big]$$

$$= \frac{-1}{4(p_1 p_2)(p_1 p_3)} \mathrm{Tr}\Big[\not{p}_4 (-\gamma^\mu \not{p}_3 \gamma^\nu + 2 p_1^\nu \gamma^\mu) \not{p}_1 (\gamma_\mu \not{p}_2 \gamma_\nu + 2 p_{1\mu} \gamma_\nu) + m_e^2 (-\gamma^\mu \not{p}_3 \gamma^\nu + 2 p_1^\nu \gamma^\mu)(\gamma_\mu \not{p}_2 \gamma_\nu + 2 p_{1\mu} \gamma_\nu) \Big]$$

$$= \frac{-1}{4(p_1 p_2)(p_1 p_3)} \mathrm{Tr}\Big[-\not{p}_4 \gamma^\mu \not{p}_3 \gamma^\nu \not{p}_1 \gamma_\mu \not{p}_2 \gamma_\nu + 2 \not{p}_4 \gamma^\mu \not{p}_1 \gamma_\mu \not{p}_2 \not{p}_1 - 2 \not{p}_4 \not{p}_1 \not{p}_3 \gamma^\nu \not{p}_1 \gamma_\nu + 4 \not{p}_4 \not{p}_1 \not{p}_1 \not{p}_1 + m_e^2 \big(-\gamma^\mu \not{p}_3 \gamma^\nu \gamma_\mu \not{p}_2 \gamma_\nu + 2 \cdot 4 \not{p}_2 \not{p}_1 - 2 \not{p}_1 \not{p}_3 \cdot 4 + 4 \not{p}_1 \not{p}_1 \big) \Big]$$

$$= \frac{-1}{4(p_1 p_2)(p_1 p_3)} \mathrm{Tr}\Big[2 \not{p}_4 \not{p}_1 \gamma^\nu \not{p}_3 \not{p}_2 \gamma_\nu - 4 \not{p}_4 \not{p}_1 \not{p}_2 \not{p}_1 + 4 \not{p}_4 \not{p}_1 \not{p}_3 \not{p}_1 + 4 \not{p}_4 \not{p}_1 m_e^2 + m_e^2 \big(-4 \not{p}_2 \not{p}_3 + 8 \not{p}_2 \not{p}_1 - 8 \not{p}_1 \not{p}_3 + 4 m_e^2 \big) \Big]$$

$$= \frac{-1}{4(p_1 p_2)(p_1 p_3)} \mathrm{Tr}\Big[8 \not{p}_4 \not{p}_1 \, p_2 \cdot p_3 - 4 \not{p}_4 \not{p}_1 \not{p}_2 \not{p}_1 + 4 \not{p}_4 \not{p}_1 \not{p}_3 \not{p}_1 + 4 \not{p}_4 \not{p}_1 m_e^2 + m_e^2 \big(-4 \not{p}_2 \not{p}_3 + 8 \not{p}_2 \not{p}_1 - 8 \not{p}_1 \not{p}_3 + 4 m_e^2 \big) \Big]$$

$$= \frac{-1}{4(p_1 p_2)(p_1 p_3)} \Big(32 (p_1 p_4 \, p_2 p_3) - 16 (p_4 \cdot p_1 \, p_2 \cdot p_1 - p_2 \cdot p_4 m_e^2 + p_1 \cdot p_4 \, p_1 \cdot p_2) + 16 (p_4 \cdot p_1 \, p_3 \cdot p_1 - p_3 \cdot p_4 m_e^2 + p_1 \cdot p_4 \, p_1 \cdot p_3) + 16 m_e^2 p_1 p_4 - 16 p_2 \cdot p_3 m_e^2 + 32 m_e^2 p_1 p_2 - 32 p_1 p_3 m_e^2 + 16 m_e^4 \Big)$$

$$= \frac{-1}{(p_1\cdot p_2)(p_1\cdot p_3)}\Big(8p_1\cdot p_4\, p_2\cdot p_3 - 8p_1\cdot p_2\, p_1\cdot p_4 + 8p_1\cdot p_3\, p_1\cdot p_4 + 4m_e^2(p_2\cdot p_4 - p_3\cdot p_4 + p_1\cdot p_4 - p_2\cdot p_3 + 2p_1\cdot p_2 - 2p_1\cdot p_3) + 4m_e^4\Big)$$

Since $s = m_e^2 + 2p_1\cdot p_2 = m_e^2 + 2p_3\cdot p_4 \Rightarrow p_3\cdot p_4 = p_1\cdot p_2$

Also $t = m_e^2 - 2p_1\cdot p_3 = m_e^2 - 2p_2\cdot p_4 \Rightarrow p_1\cdot p_3 = p_2\cdot p_4$

but $u = 2m_e^2 - 2p_1\cdot p_4 = -2p_2\cdot p_3.$

$\Rightarrow m_e^2 - p_1\cdot p_4 = -p_2\cdot p_3, \quad p_1\cdot p_4 - p_2\cdot p_3 = m_e^2$

$$= \frac{-1}{p_1\cdot p_2\, p_1\cdot p_3}\Big(8p_1\cdot p_4(p_2\cdot p_3 - p_1\cdot p_2 + p_1\cdot p_3 + p_1\cdot p_4 - p_1\cdot p_4) + 4m_e^2(-p_1\cdot p_3 + p_1\cdot p_2) + 8m_e^4\Big)$$

$$= \frac{-1}{p_1\cdot p_2\, p_1\cdot p_3}\Big(8p_1\cdot p_4(p_1\cdot(p_3+p_4-p_2)) - 8m_e^2\, p_1\cdot p_4 + 4m_e^2(-p_1\cdot p_3 + p_1\cdot p_2) + 8m_e^4\Big)$$

$$= \frac{-1}{p_1\cdot p_2\, p_1\cdot p_3}\Big(4m_e^2(-p_1\cdot p_3 + p_1\cdot p_2) + 8m_e^4\Big) \quad ✓$$

Third trace

$$\frac{-1}{4p_1\cdot p_2\, p_1\cdot p_3}\mathrm{Tr}\Big[\not{p}_4(\gamma^\nu\not{p}_2\gamma^\mu + 2p_1^\mu\gamma^\nu)\not{p}_1(-\gamma^\nu\not{p}_3\gamma^\mu + 2p_1^\nu\gamma^\mu) + m_e^2(\gamma^\nu\not{p}_2\gamma^\mu + 2p_1^\mu\gamma^\nu)(-\gamma^\nu\not{p}_3\gamma^\mu + 2p_1^\nu\gamma^\mu)\Big]$$

$$= \frac{-1}{4p_1\cdot p_2\, p_1\cdot p_3}\mathrm{Tr}\Big[+2\not{p}_4\gamma^\nu\not{p}_2\not{p}_3\gamma^\nu\not{p}_1 - 4\not{p}_4\not{p}_1\not{p}_2\not{p}_1 + 4\not{p}_4\not{p}_1\not{p}_3\not{p}_1 + 4\not{p}_4\not{p}_1 m_e^2 + m_e^2(-4\not{p}_3\not{p}_2 - 8\not{p}_3\not{p}_1 + 8\not{p}_1\not{p}_2 + 4m_e^2)\Big]$$

$$= \frac{-1}{4p_1\cdot p_2\, p_1\cdot p_3}\Big(32p_1\cdot p_4\, p_2\cdot p_3 - 16(p_1\cdot p_4\, p_2\cdot p_1 - p_2\cdot p_4 m_e^2 + p_1\cdot p_4\, p_1\cdot p_2) + 16(p_1\cdot p_4\, p_1\cdot p_3 - p_4\cdot p_3 m_e^2 + p_1\cdot p_3\, p_1\cdot p_4) + 16m_e^2\, p_1\cdot p_4 - 16p_2\cdot p_3 m_e^2 - 32p_1\cdot p_3 m_e^2 + 32p_1\cdot p_2 m_e^2 + 16m_e^4\Big)$$

$$= \frac{-1}{p_1\cdot p_2\, p_1\cdot p_3}\Big(8p_1\cdot p_4(p_2\cdot p_3 - p_1\cdot p_2 + p_1\cdot p_3) + 4m_e^2(p_2\cdot p_4 - p_3\cdot p_4 + p_1\cdot p_4 - p_2\cdot p_3 - 2p_1\cdot p_3 + 2p_1\cdot p_2) + 4m_e^4\Big)$$

* Same as second trace!

$$= \frac{-1}{p_1\cdot p_2\; p_1\cdot p_3}\left(4m_e^2(-p_1\cdot p_3 + p_1\cdot p_2) + 8m_e^4\right)$$

Last trace:

$$\frac{1}{4(p_1\cdot p_3)^2}\mathrm{Tr}\Big[\not{p}_4(-\gamma^\mu \not{p}_3 \gamma^\nu + 2p_1^\nu\gamma^\mu)\not{p}_1(-\gamma_\nu \not{p}_3\gamma_\mu + 2p_{1\nu}\gamma_\mu) + m_e^2(-\gamma^\mu\not{p}_3\gamma^\nu + 2p_1^\nu\gamma^\mu)(-\gamma_\nu\not{p}_3\gamma_\mu + 2p_{1\nu}\gamma_\mu)\Big]$$

$$= \frac{1}{4(p_1\cdot p_3)^2}\mathrm{Tr}\Big[-2\not{p}_4\gamma^\mu\not{p}_3\not{p}_1\not{p}_3\gamma_\mu - 2\not{p}_4\gamma^\mu\not{p}_1\not{p}_1\not{p}_3\gamma_\mu - 2\not{p}_4\gamma^\mu\not{p}_3\not{p}_1\not{p}_1\gamma_\mu + 4\not{p}_4 m_e^2\gamma^\mu\not{p}_1\gamma_\mu + m_e^2\left(0 - 2\gamma^\mu\not{p}_1\not{p}_3\gamma_\mu - 2\gamma^\mu\not{p}_3\not{p}_1\gamma_\mu + 4m_e^2\cdot 4\right)\Big]$$

$$= \frac{1}{4(p_1\cdot p_3)^2}\mathrm{Tr}\Big[4\not{p}_4\not{p}_3\not{p}_1\not{p}_3 + 4m_e^2\not{p}_4\not{p}_3 + 4\not{p}_4\not{p}_3 m_e^2 - 8m_e^2\not{p}_4\not{p}_1 + m_e^2(-8p_1\cdot p_3 - 8p_1\cdot p_3 + 16m_e^2)\Big]$$

$$= \frac{1}{4(p_1\cdot p_3)^2}\Big(16(p_3\cdot p_4\, p_1\cdot p_3 - 0 + p_3\cdot p_4\, p_1\cdot p_3) + 32m_e^2 p_3\cdot p_4 - 32m_e^2 p_1\cdot p_4 - 64m_e^2 p_1\cdot p_3 + 64m_e^4\Big)$$

$$= \frac{8}{(p_1\cdot p_3)^2}\Big(p_1\cdot p_3\, p_3\cdot p_4 + m_e^2(p_3\cdot p_4 - p_1\cdot p_4 - 2p_1\cdot p_3) + 2m_e^4\Big)$$

Total: Average over initial spins + pols. $(\frac{1}{2}\cdot\frac{1}{2})$.

$$\frac{1}{4}\sum_{\substack{\text{spins,}\\ \text{pols.}}}|\mathcal{M}|^2 = e^4\Big[\frac{1}{(p_1\cdot p_2)^2}\Big(2p_1\cdot p_2\, p_2\cdot p_4 + 2m_e^2(2p_1\cdot p_2 - p_2\cdot p_4 - p_1\cdot p_4) + 4m_e^4\Big) - \frac{1}{p_1\cdot p_2\, p_1\cdot p_3}\Big(2m_e^2(-p_1\cdot p_3 + p_1\cdot p_2) + 4m_e^4\Big) + \frac{1}{(p_1\cdot p_3)^2}\Big(2p_1\cdot p_3\, p_3\cdot p_4 + 2m_e^2(-2p_1\cdot p_3 + p_3\cdot p_4 - p_1\cdot p_4) + 4m_e^4\Big)\Big]$$

Note: last term is same as first with $p_2 \to -p_3$.