Lecture 15. ①

Quantum Field Theory

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Last time: Correlation functions, Wick's theorem, Feynman diagrams

Today: More Feynman diagrams

Recall from last, we calculated the first interaction term in the numerator of $\langle \Omega | T\{\phi(x)\phi(y)\}|\Omega\rangle$ in $\frac{\lambda}{4!}\phi^4$ theory.

We had

$$\langle \Omega | T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0 | T\{\phi_I(x)\phi_I(y)\exp[-i\int_{-T}^{T} dt\, H_{int}^I(t)]\}|0\rangle}{\langle 0 | T\{\exp[-i\int_{-T}^{T} dt\, H_{int}^I(t)]\}|0\rangle}$$

where $\exp[-i\int_{-T}^{T} dt \int d^3z\, \frac{\lambda}{4!}\phi(z)^4]$

$$= \exp[-i\int d^4z\, \frac{\lambda}{4!}\phi(z)^4] \simeq 1 - i\int d^4z \left(\frac{\lambda}{4!}\right)\phi(z)^4 \ldots$$

So

$$\langle \Omega | T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \langle 0 | T\{\phi_I(x)\phi_I(y)\}|0\rangle$$
$$- i\frac{\lambda}{4!}\langle 0 | T\{\phi_I(x)\phi_I(y)\int d^4z\, \phi(z)^4\}|0\rangle$$
$$+ (-i)^2\left(\frac{\lambda}{4!}\right)^2 \langle 0 | T\{\phi_I(x)\phi_I(y)\int d^4z_1\, \phi(z_1)^4 \int d^4z_2\, \phi(z_2)^4\}|0\rangle$$
$$+ \ldots$$

Since $\langle 0 | \phi^n |0\rangle$ vanishes except when all ϕ are contracted together, we developed normal ordering + Wick's theorem to help calculate.

The term linear in λ gave 2 distinct ways to contract all fields:

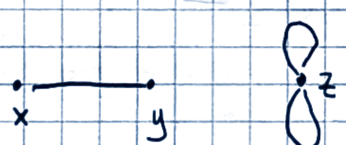
$$-i\frac{\lambda}{4!}\langle 0|\, \phi_I(x)\,\phi_I(y) \int d^4z\; \phi(z)\phi(z)\phi(z)\phi(z)\;|0\rangle$$

and

$$-i\frac{\lambda}{4!}\;\langle 0|\;\phi_I(x)\,\phi_I(y)\int d^4z\;\phi(z)\phi(z)\,\phi(z)\,\phi(z)\;|0\rangle$$

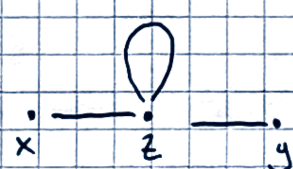
The first set had a multiplicity of 3, since we had 3 ways to pairwise contract $\phi(z)^4$ with themselves and one possibility to contract $\phi(x)$ and $\phi(y)$. The second set had a multiplicity of 12, with 4 choices for $\phi(x)$ to contract $\phi(z)^4$, ~~and~~ another 3 choices for $\phi(y)$ to contract $\phi(z)^3$, and then one pairwise contraction for $\phi(z)^2$.

Diagramatically, the first looks like:



$$\left(-i\frac{\lambda\, 3}{4!}\right) = -\frac{i\lambda}{8}\; D_F(x-y)\int d^4z\; D_F(z-z)^2$$

& the second is



$$\left(-i\frac{\lambda}{4!}\cdot 12\right) = -\frac{i\lambda}{2}\int d^4z\, D_F(x-z)\, D_F(y-z)\, D_F(z-z)$$

The total amplitude at this order is the sum of these contributions.

We can see that there is generally a counterbalance between the exponential expansion, the "normalization" of the coupling constant, and the multiplicity factor of equivalent contractions. This is encoded as a <u>symmetry factor</u>, which is the final division of the diagram by a number, taking into account these effects.

At any given order in the exponential of H_{int}^{I}, we have $\frac{1}{n!}$ for the λ^n power. Then, we also have n sets of $\int d^4 z_i$ integrals, which are all equivalent regarding a combinatorial factor, providing an $n!$ factor. For a given diagram, we get a symmetry factor of 2 for lines that start + end on the same vertex, another factor of 2 for interchange of separate propagators, and a final factor of 2 for equivalent vertices. Intuitively, the symmetry factor is the number of ways the components of a diagram can be interchanged. In practice, it is sometimes easier to perform the contractions by hand, especially if the coupling constant has an arbitrary normalization.

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We have seen that fully contracted diagrams give the only contribution to the numerator of the RHS. This is also true for the denominator, where H_{int}^{I} must contract to itself or otherwise it annihilates the vacuum states.

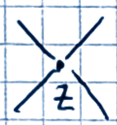
For the numerator, we sum all possible diagrams with two external points (at $\phi(x)$ + $\phi(y)$.).

We therefore can read off the diagram using Feynman rules (in position space) to determine the contribution to the amplitude.

In position space, ~~we~~ for ϕ^4 theory, we have

1. Propagators give $D_F(a-b)$

2. Vertices give $-i\lambda \int d^4z$



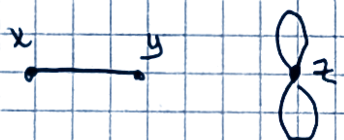
3. Each external point $= 1$

4. Divide by the symmetry factor.

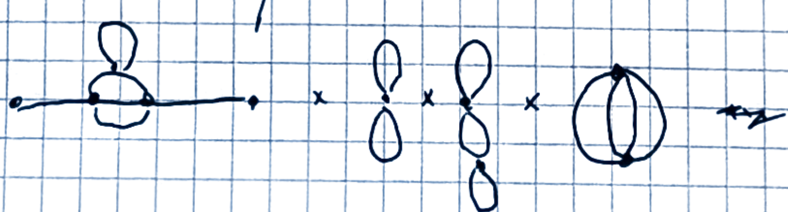
We build Feynman diagrams out of propagators, vertices, external lines, and symmetry factors, reflecting the requirement that the correlation fcn. must be <u>fully</u> <u>contracted</u>.

We can also recognize that generally, the denominator will cancel the disconnected diagrams from the numerator.

Diagramatically, we had, at $O(\lambda)$.



An arbitrary term like this has



We can rewrite schematically as

$$(\text{connected piece}) \cdot \prod_i \frac{1}{n_i!} (V_i)^{n_i}$$

where V_i is the set of "vacuum bubbles" with value V_i, and $\frac{1}{n_i!}$ is the symmetry factor of exchanging the n_i copies of V_i (which is also the reason for the n_i exponent).

The numerator is then

$$\sum_{\substack{\text{all connected}\\ \text{pieces}}} \sum_{\text{all } \{n_i\}} \begin{pmatrix}\text{connected}\\ \text{piece}\end{pmatrix} \cdot \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)$$

$\text{all } \{n_i\}$ = all ordered sets $\{n_1, n_2, n_3, \ldots\}$ of nonnegative integers

$$= \left(\sum \text{connected}\right) \cdot \sum_{\text{all}\{n_i\}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)$$

$$= \left(\sum \text{connected}\right) \left(\sum_{n_1} \frac{1}{n_1!} (V_1)^{n_1} \right) \left(\sum_{n_2} \frac{1}{n_2!} (V_2)^{n_2} \right) \cdots$$

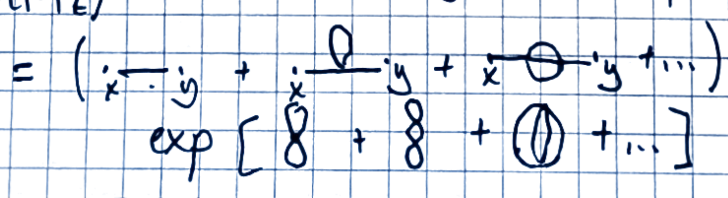
$$= \left(\sum \text{connected}\right) \prod_i \left(\sum_{n_i} \frac{1}{n_i!} V_i^{n_i} \right)$$

$$= \left(\sum \text{connected}\right) \prod_i \exp(V_i)$$

$$= \left(\sum \text{connected}\right) \exp\left(\sum_i V_i \right)$$

This is a general rule, where all vacuum bubbles exponentiate out separately from the connected pieces. So, the numerator is

$$\lim_{T \to \infty(1-i\epsilon)} \langle 0 | T\left\{ \phi_I(x) \phi_I(y) \exp\left[-i \int_{-T}^{T} dt\, H_I(t) \right] \right\} | 0 \rangle$$



But the exponentiated vacuum bubbles are exactly the denominator of the correlation function!

$$\langle 0 | T\exp\left\{ -i\int_{-T}^{T} dt\, H_I(t) \right\} | 0 \rangle = \exp[$$  $$+ \dots]$$

Intuitively, you still need fully contracted diagrams, but without $\phi(x)$ & $\phi(y)$ to contract against, you only get vacuum bubbles.

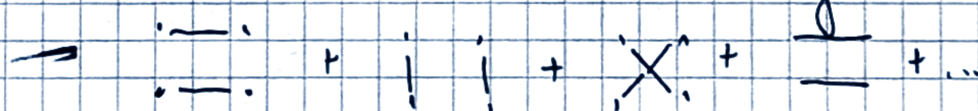
Thus, we can reexpress the original correlation function as

$$\langle \Omega | T\{\phi(x)\phi(y)\} | \Omega \rangle$$

= sum of all connected diagrams with two external points x & y.

At higher numbers of correlation fcns, we will have simply connected diagrams, disconnected diagrams, & vacuum bubbles. The vacuum bubbles exponentiate out & cancel, but the disconnected ones do not.

Ex: $\langle \Omega | T\{\phi_1 \phi_2 \phi_3 \phi_4\} | \Omega \rangle$



disconnected, but not vacuum. (x_1, x_2, x_3, x_4 to each other, but not pure $z \leftrightarrow z$).

Comments:

We can further interpret the vacuum bubbles as the vacuum energy density. (see p. 98 of P&S).

The generalization to n-fields is

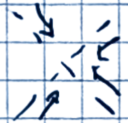
$$\langle \Omega | T\{\phi(x_1)\dots\phi(x_n)\} | \Omega \rangle = \text{sum of all connected diagrams with } n \text{ ext. pts.}$$

The $\lim_{T\to\infty(1-i\epsilon)}$ aligns with our Feynman prescription for the propagator. This is best seen in the <u>momentum</u>-space Feynman rules.

1.) 

$\frac{i}{p^2-m^2+i\epsilon}$

Arrows off line indicate momentum flow

2.) 
$-i\lambda$

3.) 

$e^{-ip_a\cdot x}$

4.) Momentum conservation at every vertex

$$(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4)$$

all flowing in

5.) Integrate over undetermined momenta

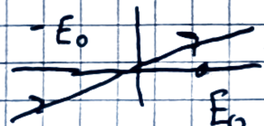
6.) Divide by symmetry factor

Note 

$$= \int d^4z\, e^{-i(p_1+p_2+p_3+p_4)\cdot z}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \int_{-T}^{T} dz^0 \int d^3z\, e^{-i(p_1+p_2+p_3+p_4)\cdot z}$$

We have this ill-defined for $|z^0|\to\infty$ unless we get an imaginary piece to the argument & then we get $e^{-|\arg|\cdot\infty}$, which converges. This justifies the contour choice of Feynman propagator:



$p^0 \propto E^0(1+i\epsilon)$

Note: we later pull out an overall momentum conservation of the Feynman diagram in momentum space, which will change rule 3 to just a factor 1.

We now have the formalism to calculate & draw Feynman diagrams. Next time, we will relate correlation fns. to the S-matrix & introduce kinematics of scattering, cross sections, & decay widths.