Lecture 14.

Quantum Field Theory 1.

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Last time:

Today: Correlation functions

We now aim to develop a formalism that allows us to visualize perturbation theory as a spacetime process, which involves developing Feynman diagrams.

Start with 2 pt. correlation fcn:

In interacting theory, we have to allow for the vacuum state to be modified compared to the free theory vacuum state.

Denote $|\Omega\rangle$ = interacting vacuum state
$|0\rangle$ = free vacuum state.

These are generally distinct objects, since interacting theories allow for non-conservation of particle number.

Recall that the 2 pt. correlation fcn. in free theory was solved:

$$\langle 0| T\{\phi(x)\phi(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} \cdot \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

$$= \underbrace{D_F(x-y)}_{\text{Feynman propagator}},$$

The two-pt. corr. fcn. in interacting theory is then

$$\langle \Omega | T\{\phi(x)\phi(y)\}|\Omega\rangle.$$

For concreteness, let us consider $\frac{\lambda}{4!}\phi^4$ theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

②

Then we must consider the interaction picture

$$\phi(x) \equiv \phi_H(x) = U^\dagger(t,t_0)\,\phi_I(t,\vec{x})\,U(t,t_0)$$

and $\quad U(t,t_0) = T\exp\left[-i\int_{t_0}^{t} dt'\; H_{int}^{I}(t')\right]$

and $\quad |\Omega\rangle \simeq |0\rangle$.

needs an important prefactor

Generally, $|\Omega\rangle$ is defined for $H_{full}|\Omega\rangle = E_\Omega|\Omega\rangle$.

So $\quad U^\dagger(t,t_0)|0\rangle = e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}|0\rangle$

$$= e^{iH(t-t_0)}\,e^{-iE_0(t-t_0)}|0\rangle$$

$$= e^{i(H-E_0)(t-t_0)}|0\rangle = \sum e^{i(H-E_0)(t-t_0)}|n\rangle\langle n|0\rangle$$

insert complete set of states

$$= e^{i(E_\Omega-E_0)(t-t_0)}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} e^{i(E_n-E_0)(t-t_0)}|n\rangle\langle n|0\rangle$$

By definition, $E_n > E_\Omega$, so we take $t\to-\infty(1-i\epsilon)$,

ensuring that $e^{i(E_n-E_0)\infty(-1+i\epsilon)} \sim e^{-\epsilon\infty(E_n-E_0)}$,

which necessarily goes faster to zero compared to the E_Ω term.

So, $\quad \lim_{t\to-\infty(1-i\epsilon)} e^{i(E_\Omega-E_0)(t-t_0)}|\Omega\rangle\langle\Omega|0\rangle = U^\dagger(t,t_0)|0\rangle$

$$\Rightarrow |\Omega\rangle = \lim_{t\to-\infty(1-i\epsilon)}\left(e^{i(E_\Omega-E_0)(t-t_0)}\langle\Omega|0\rangle\right)^{-1} U^\dagger(t,t_0)|0\rangle$$

Can also write $E_\Omega = E_\Omega - E_0$, which is the vacuum energy relative to E_0.

Interpretation: Up to a (constant) prefactor, we get the true vacuum by starting with $|0\rangle$ at $t=-\infty$ + evolving to time t_0. Necessarily need $\langle\Omega|0\rangle$ non-zero (small pert.)

We also have, analogously,

$$\langle\Omega| = \lim_{t\to+\infty(1-i\epsilon)} \left(e^{-i\tilde{E}_\Omega (t-t_0)}\langle 0|\Omega\rangle\right)^{-1}\langle 0|\,U(t,t_0)$$

Thus, we can assemble the 2-pt. corr. fn.:

for $x^0 > y^0$,

$$\langle\Omega|\,T\{\phi(x)\,\phi(y)\}\,|\Omega\rangle$$

$$= \langle\Omega|\,\phi(x)\,\phi(y)\,|\Omega\rangle$$

with full interactions, $\phi(x)=\phi_H(x)$

$$= \langle\Omega|\,\phi_H(x)\,\phi_H(y)\,|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-i\tilde{E}_\Omega T}\langle 0|\Omega\rangle\right)^{-1}\left(e^{-i\tilde{E}_\Omega T}\langle\Omega|0\rangle\right)^{-1}$$

$$\langle 0|\,U(T,t_0)\,U^\dagger(x^0,t_0)\,\phi_I(x)\,U(x^0,t_0)\,U^\dagger(y^0,t_0)\,\phi_I(y)\,U(y^0,t_0)\,U^\dagger(-T,t_0)\,|0\rangle$$

change to $-T$

$$\cdot\left(\underbrace{\cancel{e^{+i\tilde{E}_\Omega t_0}\,e^{-i\tilde{E}_\Omega t_0}}}_{1}\right)^{-1}$$

We recognize $U(T,t_0)\,U^\dagger(x^0,t_0) = U(T,x^0)$,

so

$$= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-i2\tilde{E}_\Omega T}\,|\langle 0|\Omega\rangle|^2\right)^{-1}$$

$$\langle 0|\,U(T,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y^0)\,U(y^0,-T)\,|0\rangle$$

Given that $1 = \langle\Omega|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-i2\tilde{E}_\Omega T}|\langle 0|\Omega\rangle|^2\right)^{-1}$

$$\langle 0|\,\underbrace{U(T,t_0)\,U^\dagger(-T,t_0)}_{U(T,-T)}\,|0\rangle$$

$\Rightarrow$ we divide + reexpress the prefactor as

$$\langle \Omega | \phi(x)\phi(y) | \Omega \rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| U(T,x^0)\,\phi_I(x)\, U(x^0,y^0)\,\phi_I(y)\, U(y^0,-T)|0\rangle}{\langle 0| U(T,-T)|0\rangle}$$

Note that we can also consider $y^0 > x^0$, where we would get $\langle 0| U(T,y^0)\,\phi_I(y)\, U(y^0,x^0)\,\phi_I(x)\, U(x^0,-T)|0\rangle$. in the numerator. Hence, the time ordering operation is preserved and we can write, recalling

$$U(t,t_0) \equiv T\left\{ \exp\left[-i\int_{t_0}^{t} dt'\; H_{int}^{I}(t') \right] \right\}$$

$$\langle \Omega | T\{\phi(x)\phi(y)\} | \Omega \rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\{\phi_I(x)\,\phi_I(y) \exp[-i\int_{-T}^{T} dt\; H_{int}^{I}(t)]\}|0\rangle}{\langle 0| T\{\exp[-i\int_{-T}^{T} dt\; H_{int}^{I}(t)]\}|0\rangle}$$

This readily generalizes to n-pt. correlation fns.

$$\langle \Omega | T\{\phi(x_1)\phi(x_2)\dots\phi(x_n)\} | \Omega \rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\{\phi_I(x_1)\,\phi_I(x_2)\dots\phi_I(x_n) \exp[-i\int_{-T}^{T} dt\; H_{int}^{I}(t)]\}|0\rangle}{\langle 0| T\{\exp[-i\int_{-T}^{T} dt\; H_{int}^{I}(t)]\}|0\rangle}$$

Two key comments:

1. Exact result. Can be expanded in RHS as perturbation series in H_{int}^{I}.
2. We have reduced the complexity of interacting fields as dependent on $\phi_I(x)$ + $|0\rangle$ of free theory.

To see this in action, we expand the numerator, using $H^I_{int}(t) = \int d^3z \left(\frac{+\lambda}{4!}\right)\phi(z)^4$

Then we get (all fields in interaction picture)

$$\langle 0| T\left\{ \phi(x)\phi(y) + \phi(x)\phi(y)\left(-i\int dt \int d^3z \frac{\lambda}{4!}\phi(z)^4\right) + \dots \right\}|0\rangle$$

$$= \langle 0| T\{ \phi(x)\phi(y)\}|0\rangle$$
$$-\frac{i\lambda}{4!}\langle 0| T\left\{ \phi(x)\phi(y)\int d^4z\, \phi(z)\phi(z)\phi(z)\phi(z)\right\}|0\rangle + \dots$$

How do we evaluate the second term?
Recall the first term is simply $D_F(x-y)$ (Feynman propagator).

For the second term, we can always perform the simplification that $a_p|0\rangle = 0$ and $\langle 0|a^+ = 0$.

To that end, we can split up

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

for $\phi^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} a_p e^{-ip\cdot x}$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} a_p^+ e^{+ip\cdot x}$$

So $\phi^+(x)|0\rangle = 0$ + $\langle 0|\phi^-(x) = 0$

Then, explicitly, for $x^0 > y^0$,

$$\begin{aligned}
T\{\phi(x)\phi(y)\} &= \phi(x)\phi(y) \\
&= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) \\
&\quad + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \\
&= \phi^+(x)\phi^+(y) + [\phi^+(x),\phi^-(y)] + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y)
\end{aligned}$$

Note that all terms except for the commutator have all a_p to the right of all $a_q^\dagger$, and so they have vanishing vacuum expectation value. Any product written this way is called "normal order," and we can define

$$N(a_p a_k^\dagger a_q) = a_k^\dagger a_p a_q \text{ , for example}$$
$$= a_k^\dagger a_q a_p .$$

So for $x^0 > y^0$,

$$\langle 0| T\{\phi(x)\phi(y)\}|0\rangle = \langle 0|[\phi^+(x), \phi^-(y)]|0\rangle$$

Similarly, for $x^0 < y^0$, we get

$$\langle 0| T\{\phi(x)\phi(y)\}|0\rangle = \langle 0|[\phi^+(y), \phi^-(x)]|0\rangle$$

We can define the "Wick contraction"

$$\overbrace{\phi(x)\,\phi}(y) = \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y^0 > x^0 \end{cases}$$
$$= D_F(x-y).$$

and we reproduce $\langle 0| T\{\phi(x)\phi(y)\}|0\rangle = D_F(x-y)$ as before.

Using the normal ordering operator, we can relate it to time ordering as

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y) + \overbrace{\phi(x)\,\phi}(y)\}$$

since $\langle 0| N\{\phi(x)\phi(y)\}|0\rangle = 0$ were all terms that vanished.

This also generalizes to <u>Wick's theorem</u>

$$T\{\phi(x_1)\ldots\phi(x_n)\} = N\{\phi(x_1)\ldots\phi(x_n) + \text{all possible contractions of } n \text{ fields in pairs}\}$$

Explicitly, with $\phi(x_i) \equiv \phi_i$,

$$T\{\phi_1\phi_2\phi_3\phi_4\} = N\{\phi_1\phi_2\phi_3\phi_4 + \overbrace{\phi_1\phi_2}\phi_3\phi_4 + \overbrace{\phi_1\phi_2\phi_3}\phi_4$$
$$+ \overbrace{1\,2\,3\,4} + 1\,\overbrace{2\,3}\,4 + 1\,\overbrace{2\,3\,4} + 1\,2\,\overbrace{3\,4}$$
$$+ \overbrace{1\,2}\,\overbrace{3\,4} + \overbrace{1\,\underbrace{2\,3}\,4} + \overbrace{1\,\underbrace{2\,3}}\,4\}$$

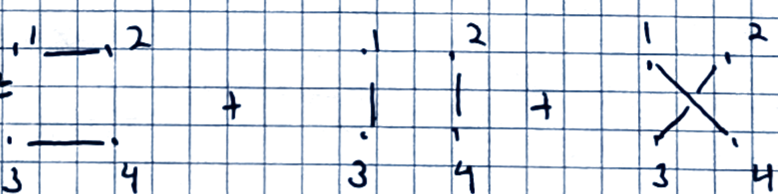
Note $N\{\overbrace{\phi_1\phi_2\phi_3\phi_4}\} = D_F(x_1 - x_4)\, N\{\phi_2\phi_3\}$

Now, sandwich $N\{\ldots\}$ inside inner product with vacuum states: the only nonzero terms are those with all fields contracted, so we get

$$\langle 0| T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = \langle 0| N\{\overbrace{1\,2}\,\overbrace{3\,4} + \overbrace{1\,\underbrace{2\,3}\,4} + \overbrace{1\,\underbrace{2\,3}\,4}\}|0\rangle$$
$$= D_F(x_1 - x_2)\, D_F(x_3 - x_4)$$
$$+ D_F(x_1 - x_3)\, D_F(x_2 - x_4)$$
$$+ D_F(x_1 - x_4)\, D_F(x_2 - x_3).$$

This is best visualized using Feynman diagrams, where we can label four spacetime points $x_1 \ldots x_4$ & draw all pairwise connections:

$\langle 0| T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle =$



For the interaction term,

$$\langle 0| T\{\phi(x)\,\phi(y) \left(\frac{-i\lambda}{4!}\right) \int d^4z\, (\phi(z))^4\}|0\rangle$$
$$= \frac{-i\lambda}{4!} \langle 0| N\{\phi(x)\,\phi(y) \int d^4z\, (\phi(z))^4 + \text{all possible contractions}\}|0\rangle$$

We have two distinct possibilities to get full contractions.

$$\phi(x)\,\phi(y)\ \Big(\phi(z)\ \phi(z)\ \phi(z)\ \phi(z)\Big) = 3 \text{ choices}$$

as 1 2 3 4, 1 2 3 4, 1 2 3 4

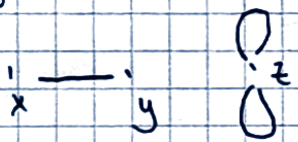
or x to z (4), y to z (3), z to z (1) = 12 choices

$$\Rightarrow \langle 0|\, T\left\{ \phi(x)\phi(y) \left(\frac{-i\lambda}{4!}\right) \int d^4z\, (\phi(z))^4 \right\} |0\rangle$$

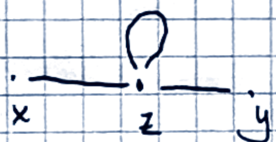
$$= \frac{-i\lambda}{4!}\, 3\, D_F(x-y) \int d^4z\, D_F(z-z)\, D_F(z-z)$$

$$+ 12 \left(\frac{-i\lambda}{4!}\right) \int d^4z\, D_F(x-z)\, D_F(y-z)\, D_F(z-z)$$

In diagrams, first is



, while second is



We will define the corresponding Feynman rules for these Feynman diagrams next time.

Proof of Wick's theorem: Follows by induction.

For $m-1$ fields, in time order,

$$T\{\phi_1 \dots \phi_m\} = \phi_1 \dots \phi_m$$

$$= (\phi_1^+ + \phi_1^-)\, N\{\phi_2 \dots \phi_m + (\text{all contractions not involving } \phi_1)\}$$

Term with ϕ_1^+ needs to combinatorially contract to move inside normal ordering.

See P+S, eqn. 4.41.