Lecture 9. Massless vs. massive fermions

1-11-19. Gravitational anomalies, Witten's $SU(2)$ anomaly,

Lecture 10: Scale anomaly / Weyl anomaly.
QCD, chiral symmetry breaking, $\pi^0 \to \gamma\gamma$.
+ Chiral Effective Lagrangians.

Consider two phases of SM gauge symmetry.

Broken phase: $SU(3)_c \times U(1)_{em}$

Unbroken phase: $SU(3)_c \times SU(2)_W \times U(1)_Y$

In broken phase, all SM charged fermions are in vector-like representations under color + EM. Moreover, SM fermions are all massive:

Exercise: $\mathcal{L} \supset y_u \bar{Q}_L \tilde{H} u_R + y_d \bar{Q}_L H d_R + y_e \bar{L}_L H e_R + h.c.$

$\Rightarrow = m_u^{diag} \bar{u} u + m_d^{diag} \bar{d} d + m_e^{diag} \bar{e} e$

(Note no LH or RH projections in broken phase.)

Note this is exemplar of the connection between vector gauge theories allowing tree-level masses + chiral gauge theories forbidding tree-level masses.

Another remark: you can, as a model-building tool, always promote tree-level masses to chiral / Yukawa masses by replacing $m \to y \langle\phi\rangle$ for some $\langle\phi\rangle$ vev, where ϕ is a scalar field that breaks an axial symmetry.

Example: $m_N \bar{N}^c N \to y \phi \bar{N}_L^c N_R + h.c.$

for Majoron field ϕ.

Now, consider calculations of triangle diagrams using massive fermions.

First, consider unbroken vectors on external vertices.

Four possibilities: ggg, $gg\gamma$, $g\gamma\gamma$, $\gamma\gamma\gamma$.

Note ggg is exactly the same anomaly calculation as the unbroken $SU(3)^3$: divergence in mass-independent matrix element vanishes since quarks are in vectorlike reps.

Separately, $gg\gamma$ divergence vanishes since quarks are vectorlike. (Mass-dependent piece should also vanish from Furry's theorem.) $g\gamma\gamma$ vanishes since $SU(3)$ generators are traceless. Finally, $\gamma\gamma\gamma$ vanishes by Furry's theorem.

What about massive vectors on external vertices?

ggZ has VL quarks under $SU(3)$. Also note that up quarks + down quarks have opposite axial charges, so the ggZ divergence vanishes when summing over all quarks.

gZZ + $gZ\gamma$ vanishes since $\mathrm{Tr}[t^a] = 0$.

The vanishing of divergences in loops of gluons + other ext. vectors is related to the fact that $SU(3)$ is a product gauge group $\Rightarrow$ there are no counterterms at tree-level to mix gluons with other vectors. Note ggZ exists at one-loop as a vertex fcn., but if Z is on-shell, it vanishes by Landau-Yang. Moreover, vertex fcn. must arise from masses of fermions.

Now, remaining neutral current triangles are ZZZ, $ZZ\gamma$, + $Z\gamma\gamma$.

Exercise: Calculate ZZZ. Should vanish from a related form of Furry's theorem. $ZZ\gamma$ is similar. (No EM current for Z-Z interaction.)

$Z\gamma\gamma$: Most interesting. Has contributions from all SM charged fermions. The overall divergence is controlled by anomalies again, but since axial couplings are weighted by EM charge squared, the calculation is not trivial. (Exercise!) Even so, the remaining contribution comes from the mass-dependent parts. (As a decay of $Z \to \gamma\gamma$, the vertex fcn. does not vanish but $Z \to \gamma\gamma$ decay vanishes from Landau-Yang.)

Given the fact that the vertex fcns. for all of these triangle diagrams have vanishing divergences only when summing over anomaly-free sets of massive fermions, what happens to these divergence cancellations when we calculate in EFTs when these massive fermions are integrated out?

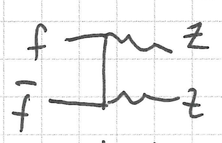
In EFT, (example, SM EFT w/o top), the remaining light fermions (bottom, charm, strange, up, down, leptons) will leave an uncancelled divergence in their mass independent loop calculation. This requires a mass independent term in the EFT, so-called Wess-Zumino term, $A\,\epsilon^{\mu\nu\rho\sigma} Z_\mu A_\nu F_{\rho\sigma}$, with coefficient of A from the top triangle anomaly contribution.

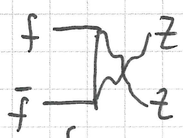
Such WZ terms are generic when integrating out chiral fermions. This can be understood since chiral fermions are massless in the unbroken phase of a theory: Integrating out "massless" fermion modes requires a corresponding compensating term to restore gauge invariance + the Ward identity conservation, hence the prefactor for the WZ term is the anomaly coeff.

Last remark on massive vs. massless fermions:

Massive fermions can only have axial vector couplings as a result of some chiral symmetry breaking.

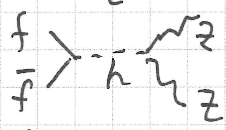
Not only ~~do~~ are sums of axial couplings ~~a~~ governed by anomaly cancellation conditions, individual axial couplings to massive gauge bosons lead to unitarity violation in $f\bar{f} \to ZZ$ scattering. (Exercise: Calculate $f\bar{f} \to ZZ$ in the SM. Should have





diagrams.

Show unitarity is restored by



, and ~~so~~ requires the equivalence $g^4 m_f^2 = y_f^2 \cdot \frac{m_Z^4}{v^2}$)

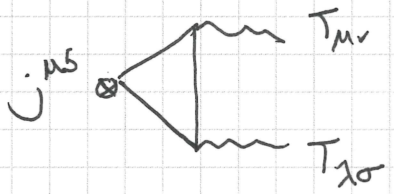
So, there is a structural connection between axial couplings to vectors, massive fermions, + anomaly cancellation.

Gravitational anomalies

Original ref. Alvarez-Gaume, Witten Nucl. Physics B, 234 (1983) 269.

Follow sect. 3. Motivate $U(1)\times G^2$ anomaly cancellation condition.

Consider triangle diagram with ext. current + two gravitons.



+ crossed

If $j^{\mu 5}$ is non-Abelian, then we get a trace over its generator & the diagram vanishes. $\Rightarrow$ no $SU(2)\times G^2$ or $SU(3)\times G^2$ anomalies.

For $U(1)$, the calculation gives

$$D_\mu j^5_\mu = \frac{-1}{384\pi^2} R\tilde{R}$$

for $R\tilde{R} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} \underbrace{R_{\mu\nu\sigma\tau}}_{\text{Rieman tensor}} R_{\alpha\beta}{}^{\sigma\tau}$

For j^5_μ coupled to a gauge field A_μ, the current non-conservation eqn. corresponds to an effective action that is not invariant under the gauge transformation $A_\mu \to A_\mu - \partial_\mu \epsilon$, giving

$$\rightarrow \delta\Gamma = \frac{-1}{384\pi^2}\int d^4x \sqrt{g}\, \epsilon\, R\tilde{R}$$

change in action

Consider coordinate transformation

$$x^\mu \to x^\mu + \epsilon^\mu$$

$$\delta g_{\mu\nu} = -D_\mu\epsilon_\nu - D_\nu\epsilon_\mu$$

Leads to $\delta\Gamma = -\int d^4x \sqrt{g}\, \frac{\delta\Gamma}{\delta g_{\mu\nu}} (D_\mu\epsilon_\nu + D_\nu\epsilon_\mu)$

Since $\frac{\delta\Gamma}{\delta g_{\mu\nu}} = \frac{1}{2}\langle T_{\mu\nu}\rangle$ + symmetric in $\mu \leftrightarrow \nu$,

$$\delta\Gamma = \int d^4x \sqrt{g}\, \epsilon_\nu D_\mu \langle T^{\mu\nu}\rangle$$

Invariant effective action under infinitesimal coordinate trans.

$\Leftrightarrow$ induced energy momentum tensor of matter fields is conserved.

Analogous to conservation of induced current for gauge invariance of effective action.

Implies $\sum_f U(1) = 0$. (weight by $U(1)$ charges.)

Aside: Can also have purely gravitational anomalies, only in $d = 4k+2$.

For d = odd, Lorentz group has one type of spinor rep.

Couplings to gravity preserve parity, give real effective action free of anomalies.

For d = even, Lorentz group has 2 spinor reps.

Can have PV gravitational couplings consistent with CPT only if $d = 4k+2$.

(CPT reverses chirality of fermions in $d = 4k$.

CPT invariance requires equal #s of + + − chirality + couplings to gravity preserve parity.

Witten's SU(2) anomaly. Phys. Lett. 117B, 5 (1982)

SU(2) gauge theory with odd # of 4d Weyl fermions in fund. rep. is inconsistent.

Aside on topology

Basic homotopy: Consider manifold M, given point x_0, closed paths γ_i, which begin + end at x_0.

Two paths are homotopic if can continuously deform into each other.

Paths define a group space with composition as group operation. $\gamma_i \circ \gamma_j$ means traverse γ_i, then γ_j.

Composition of γ_1 + γ_1', which are homotopically equivalent, and γ_2 + γ_2', which are also homotopically equivalent, gives paths $\gamma_1 \circ \gamma_2$ and $\gamma_1' \circ \gamma_2'$ which are homotopically equivalent.

Inverse of each path is traversing path in opposite direction.

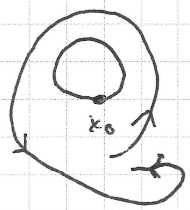
If M has only one connected component, the group corresponding to composition of paths is independent of the base point x_0. This ground is the fundamental group of M, known as the homotopy group $\pi_1(M)$.

Consider $\pi_1(S^1)$. Recall S^1 is a circle.

Identity element of group $\pi_1(S^1)$ is paths which do not wrap around the circle $\Rightarrow$ can always be deformed to paths that stay at x_0:

Nontrivial group element:

The paths that wrap around once:



Since path composition adds winding numbers, the group $\pi_1(S^1)$ is isomorphic to the integers under addition,

$$\pi_1(S^1) = \mathbb{Z}.$$

For S^n, $n \geq 2$, any path is deformable to nothing.

$$\pi_1(S^n) = 1.$$

For $\pi_1(M) = 1$, we call the space "simply connected."

End asides ↑

$\pi_n(M)$ is the group of mappings from S^n to manifold M.

It turns out that the relevant homotopy group is $\pi_4(SU(2)) = \mathbb{Z}_2$

So Euclidean space (S^3) has gauge transformation with two wrappings $\Leftrightarrow$ deform to identity.

A $U(x)$ gauge transformation, $U(x) \to 1$ as $|x| \to \infty$.

Euclidean path integral

$$\int (dA_\mu) \exp\left(\frac{-1}{2g^2} \int d^4x \, \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}\right) \quad \text{double-counting}$$

$$A_\mu^U = U^{-1} A_\mu U - i \underbrace{U^{-1} \partial_\mu U}_{\text{pure gauge}}$$

$$Z = \int d\Psi \, d\bar{\Psi} \int dA_\mu \exp\left(-\int d^4x \left[\frac{1}{2g^2} \mathrm{tr}\, F_{\mu\nu}^2 + \bar{\Psi} i\not{D} \Psi\right]\right)$$

Dirac index: $\int (d\Psi \, d\bar{\Psi})_{Dirac} \exp(\bar{\Psi} i\not{D} \Psi) = \det i\not{D}$

For Weyl fermion, Dirac fermion doublet is two LH Weyl doublets.

One Weyl doublet: $\int (d\Psi \, d\bar{\Psi})_{Weyl} \exp(\bar{\Psi} i\not{D} \Psi) = (\det i\not{D})^{1/2}$

Now, pick gauge field A_μ, can choose $(\det i\not{D})^{1/2}$ sign.

Note $(\det i\not{D})^{1/2}$ is invariant under infinitesimal gauge transformations.

Can show $[\det i\not{D}(A_\mu)]^{1/2} = -[\det i\not{D}(A_\mu^U)]^{1/2}$

$$\Rightarrow Z = \int dA_\mu \, (\det i\not{D})^{1/2} \exp\left(\frac{-1}{2g^2} \int d^4x \, \mathrm{tr}\, F_{\mu\nu}^2\right) = 0$$

Spectrum is discrete for large volume sphere.

No zero eigenvalues.
$i\not{D}$ eigenvalues real, $i\not{D}\Psi = \lambda \Psi$, then $i\not{D}(\gamma_5 \Psi) = -\lambda \gamma_5 \Psi$

So, choose + eigenvalues for particular gauge field A_μ $(\lambda, -\lambda)$. Vary A_μ to A_μ^U, $A_\mu(t) = (1-t)A_\mu + t A_\mu^U$.

One pair of eigenvalues $\{\lambda(t), -\lambda(t)\}$ crosses.

For n doublets, get $(\det i\not{D})^{n/2}$.