Lecture 6. ①

11-23-18.

Finish Fujikawa. Non-renormalization thm. by Adler-Bardeen

Bilal: Sec. 3.1, 3.2.

$$e^{i\tilde{W}[A]} = \int D\Psi D\bar{\Psi}\, e^{i\int \mathcal{L}_{matter}[\Psi, \bar{\Psi}, D_\mu\Psi, D_\mu\bar{\Psi}]}$$

Can use functional integral to calculate vacuum expectation values of time-ordered products of $\mathcal{O}$ operators involving matter + gauge fields.

Simplest to do functional integral over matter fields alone then deal with gauge fields, gauge-fixing + ghosts separately.

Aside: Peskin notation:

eq. 9.73 $$Z[\bar{\eta}, \eta] = \int D\bar{\Psi} D\Psi \exp\left[i\int d^4x\left(\bar{\Psi}(i\partial\!\!\!/ - m)\Psi + \bar{\eta}\Psi + \bar{\Psi}\eta\right)\right]$$

$\eta(x)$ is Grassmann-valued source field.

$$\langle 0 | T\Psi(x_1)\bar{\Psi}(x_2)|0\rangle = Z_0^{-1}\left(\frac{-i\,\delta}{\delta\bar{\eta}(x_1)}\right)\left(\frac{+i\,\delta}{\delta\eta(x_2)}\right)Z(\bar{\eta},\eta)\Big|_{\bar{\eta},\eta=0}$$

Consider local transformation

$$\Psi(x) \to \Psi'(x) = U(x)\Psi(x)$$

$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}\,\bar{U}$$

with $\bar{U} \equiv i\gamma^0 U^\dagger i\gamma^0$

Metric convention in Bilal is $(- + + +)$

(from $\Psi^\dagger \to \Psi^\dagger U^\dagger \gamma^0\gamma^0$
$\Rightarrow \bar{\Psi} \to \Psi^\dagger U^\dagger \gamma^0$
$\equiv \bar{\Psi}\,\bar{U}$)

Transformation of fermion measure $D\Psi + D\bar{\Psi}$:

$$D\Psi \to D\Psi' = (\mathrm{Det}\,\mathcal{U})^{-1} D\Psi$$

$$D\bar{\Psi} \to D\bar{\Psi}' = (\mathrm{Det}\,\bar{\mathcal{U}})^{-1} D\bar{\Psi}$$

$\mathcal{U}$ + $\bar{\mathcal{U}}$ operators given by $\langle x|\mathcal{U}|y\rangle = U(x)\,\delta^4(x-y)$

$\langle x|\bar{\mathcal{U}}|y\rangle = \bar{U}(x)\,\delta^4(x-y)$

First consider non-chiral transformation.

$$\mathcal{U}(x) = e^{i\epsilon^\alpha(x) t_\alpha}, \qquad t_\alpha^\dagger = t_\alpha, \quad [\gamma^\mu, t_\alpha] = 0$$

Then $\bar{\mathcal{U}}(x) = i\gamma^0 e^{-i\epsilon^\alpha(x) t_\alpha} i\gamma^0 = e^{-i\epsilon^\alpha(x) t_\alpha} (i\gamma^0)^2 = e^{-i\epsilon^\alpha(x) t_\alpha} = \mathcal{U}^{-1}(x)$

and $\bar{\mathcal{U}} = \mathcal{U}^{-1} \Rightarrow (\mathrm{Det}\,\mathcal{U})^{-1} (\mathrm{Det}\,\bar{\mathcal{U}})^{-1} = 1$

+ fermion measure is invariant, in particular fermion measure is invariant under gauge transformations.

Second, consider chiral transformation.

$$\mathcal{U}(x) = e^{i\epsilon^\alpha(x) t_\alpha \gamma^5}$$

$$\bar{\mathcal{U}}(x) = i\gamma^0 e^{-i\epsilon^\alpha(x) t_\alpha \gamma^5} i\gamma^0 = e^{+i\epsilon^\alpha(x) t_\alpha \gamma^5} (i\gamma^0)^2 = e^{+i\epsilon^\alpha(x) t_\alpha \gamma^5} = \mathcal{U}(x)$$

Then $\bar{\mathcal{U}} = \mathcal{U} \Rightarrow (\mathrm{Det}\,\mathcal{U})^{-1} (\mathrm{Det}\,\bar{\mathcal{U}})^{-1} = (\mathrm{Det}\,\mathcal{U})^{-2}$

The fermion measure is not necessarily unity + must be computed. The final dets. $\mathrm{Det}\,\mathcal{U}$ + $\mathrm{Det}\,\bar{\mathcal{U}}$ should be computed with gauge invariant regulator in order that gauge path integral is not spoiled or modified.

Can consider the action of operator $\mathcal{U}$ as

$$\langle x | \mathcal{U}^2 | y \rangle = \int d^4z \, \langle x | \mathcal{U} | z \rangle \langle z | \mathcal{U} | y \rangle = \int d^4 z \, \mathcal{U}(x) \delta^4(x-z) \cdot \mathcal{U}(z) \delta^4(z-y)$$

$$= \mathcal{U}^2(x) \delta^4(x-y)$$

$$\Rightarrow \langle x | f(\mathcal{U}) | y \rangle = f(\mathcal{U}(x)) \langle x | y \rangle$$

funct + matrix trace

$$\mathrm{Tr} \log \mathcal{U} = \int d^4x \, \langle x | \underset{\substack{\uparrow \\ \text{matrix trace}}}{\mathrm{tr}} \log(\mathcal{U}) | x \rangle$$

$$= \int d^4 x \, \delta^4(x-x) \, \mathrm{tr} \log(\mathcal{U}(x))$$

$$= \int d^4 x \, \delta^4(0) \, i\epsilon^\alpha(x) \, \mathrm{tr}\, t_\alpha \gamma_5$$

Then $(\mathrm{Det}\,\mathcal{U})^{-2} = e^{-2\mathrm{Tr} \log \mathcal{U}} = e^{i\int d^4x \, \epsilon^\alpha(x) a_\alpha(x)}$,

$$a_\alpha(x) = -2\, \delta^4(0) \, \mathrm{tr}\, t_\alpha \gamma_5$$

Now, regularize $a_\alpha(x)$. Naively, $\delta^4(0)=\infty$ + $\mathrm{tr}[t_\alpha\gamma_5]=0$,

$\delta^4(0)$ is UV divergent.

$$\delta^4(0) = \langle x|x\rangle = \int d^4p \langle x|p\rangle\langle p|x\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\Big|_{x=y}$$

Cutoff large momentum modes (also in P+S).

$$\int d^4x\ \epsilon^\alpha a_\alpha(x) = -2\,\mathrm{Tr}\,\mathcal{T}, \quad \mathcal{T} = \epsilon^\alpha(\hat{x})\gamma_5 t_\alpha$$

$$\int d^4x\ \epsilon^\alpha a_\alpha(x) = -2\lim_{\Lambda\to\infty}\mathrm{Tr}\,\mathcal{T}_\Lambda, \quad \mathcal{T}_\Lambda = \epsilon^\alpha(\hat{x})\gamma_5 t_\alpha f\left(\left(\frac{i\hat{\not{D}}}{\Lambda}\right)^2\right)$$

f is smooth fcn. with B.Cs. $f(0)=1$, $f(\infty)=0$,

$$s f'(s)\Big|_{s=0}=0, \quad s f'(s)\Big|_{s=\infty}=0.$$

Covariant Dirac operator $\langle x|\hat{\not{D}}|x\rangle = \not{D}\,\delta\langle x|x\rangle$

$$\mathrm{Tr}\,\mathcal{T}_\Lambda = \int d^4x\ \mathrm{tr}\langle x|\epsilon^\alpha(\hat{x})\gamma_5 t_\alpha f\left(\left(\frac{i\hat{\not{D}}}{\Lambda}\right)^2\right)|x\rangle$$

$$= \int d^4x\ \epsilon^\alpha(x)\int d^4p\ \langle x|p\rangle\ \mathrm{tr}\ \gamma_5 t_\alpha \langle p|f\left(\left(\frac{i\hat{\not{D}}}{\Lambda}\right)^2\right)|x\rangle$$

$$= \int d^4x\ \epsilon^\alpha(x)\int\frac{d^4p}{(2\pi)^4} e^{ip\cdot x}\ \mathrm{tr}\ \gamma_5 t_\alpha\ f\left(\frac{-1}{\Lambda^2}\left(\gamma^\mu\left(\frac{\partial}{\partial x^\mu}-iA^R_\mu(x)\right)\right)^2\right)e^{-ip\cdot x}$$

$$= \int d^4x\ \epsilon^\alpha(x)\int\frac{d^4p}{(2\pi)^4} e^{ip\cdot x}\ \mathrm{tr}\ \gamma_5 t_\alpha\ e^{-ip\cdot x} f\left(\frac{-1}{\Lambda^2}\left(\gamma^\mu\left(\frac{\partial}{\partial x^\mu}-ip_\mu-iA_\mu(x)\right)\right)^2\right)$$

$$= \int d^4x\ \epsilon^\alpha(x)\int\frac{d^4p}{(2\pi)^4}\ \mathrm{tr}\ \gamma_5 t_\alpha\ f\left(\frac{-1}{\Lambda^2}(-i\not{p}+\not{D})^2\right)$$

$$= \int d^4x\ \epsilon^\alpha(x)\ \Lambda^4\int\frac{d^4q}{(2\pi)^4}\ \mathrm{tr}\ \gamma_5 t_\alpha f\left(-\left(-i\not{q}+\frac{\not{D}}{\Lambda}\right)^2\right)$$

Expand $f\left(-\left(-i\not{q}+\frac{\not{D}}{\Lambda}\right)^2\right) = f\left(q^2+2iq^\mu D_\mu - \frac{\not{D}^2}{\Lambda^2}\right)$

in Taylor series around q^2.

For non-vanishing trace, need 4 γ.

For non-vanishing limit as $\Lambda\to\infty$, need at most 4 $\frac{\not{D}}{\Lambda}$

Picks out $\frac{1}{2} f''(q^2) \left(\frac{-\cancel{D}^2}{\Lambda^2}\right)^2$ in Taylor series:

$$\lim_{\Lambda\to\infty} \mathrm{Tr}\, \tau_\Lambda = \int d^4x\, \epsilon^\alpha(x) \int \frac{d^4q}{(2\pi)^4} \frac{1}{2} \cdot f''(q^2)\, \mathrm{tr}\, \gamma_5 t_\alpha (-\cancel{D}^2)^2$$

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{2} f''(q^2) = \frac{i}{2(2\pi)^4} \mathrm{vol}(S^3) \int_0^\infty dq\, q^3 f''(q^2) = \frac{i}{2(2\pi)^4} \cdot 2\pi^2 \frac{1}{2} \int_0^\infty d\xi$$
$$\times \xi f''(\xi)$$

$$= \frac{i}{32\pi^2} \left(\xi f'(\xi) \Big|_0^\infty - \int_0^\infty d\xi\, f'(\xi) \right)$$

$$= \frac{i}{32\pi^2}$$

$$\cancel{D}^2 = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} D_\mu D_\nu + \frac{1}{2}[\gamma^\mu,\gamma^\nu] D_\mu D_\nu = D^\mu D_\mu - \frac{i}{4}[\gamma^\mu,\gamma^\nu] F_{\mu\nu}$$

$$\mathrm{tr}\, \gamma_5 t_\alpha (-\cancel{D}^2)^2 = \left(\frac{i}{4}\right)^2 \mathrm{tr}_D\, \gamma_5 [\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]\, \mathrm{tr}_R\, t_a F_{\mu\nu} F_{\rho\sigma}$$

$$= -i\, \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}_R\, t_\alpha F_{\mu\nu} F_{\rho\sigma}$$

$$\Rightarrow \lim_{\Lambda\to\infty} \mathrm{Tr}\, \tau_\Lambda = \frac{1}{32\pi^2} \int d^4x\, \epsilon^\alpha \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}_R\, t_\alpha F_{\mu\nu} F_{\rho\sigma}$$

and $a_\alpha(x) = \frac{-1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma}\, \mathrm{tr}_R\, t_\alpha F_{\mu\nu}(x) F_{\rho\sigma}(x)$

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In P+S,

$$\mathcal{D}\psi' \mathcal{D}\bar\psi' = \mathcal{J}^{-2} \mathcal{D}\psi \mathcal{D}\bar\psi \qquad (19.69)$$

$$\mathcal{J} = \det(1+C) = \exp(\mathrm{tr}\log(1+C)) = \exp\left(\sum_n C_{nn} + \dots\right]$$

for eigenvalues $a_m' = \sum_n (\delta_{mn} + C_{mn}) a_n$

under γ^5 transformation

Regularizing with $e^{(i\cancel{D})^2/M^2}$ gives

$$a(x) \approx \lim_{M\to\infty} \langle x | \mathrm{tr}\left(\gamma^5 e^{(i\cancel{D})^2/M^2}\right) | x \rangle$$

Then $(i\not{D})^2 = -D^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}$ $\qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

Again, expand to order $(\sigma \cdot F)^2$ for nonzero trace + nonvanishing as $M \to \infty$:

$$\lim_{M\to\infty} \langle x| \mathrm{tr}\left(\gamma^5 e^{(-D^2 + \frac{e}{2}\sigma\cdot F)/M^2}\right)|x\rangle$$

$$= \lim_{M\to\infty} \mathrm{tr}\left[\gamma^5 \frac{1}{2!}\left(\frac{e}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^2\right] \langle x| e^{-\partial^2/M^2}|x\rangle$$

Wick rotation $\langle x| e^{-\partial^2/M^2}|x\rangle = \lim_{x\to y}\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} e^{k^2/M^2}$

$$= i\int \frac{d^4k_E}{(2\pi)^4} e^{-k_E^2/M^2}$$

$$= \frac{iM^4}{16\pi^2}$$

Gives $\lim_{M\to\infty} \frac{-ie^2}{8\cdot 16\pi^2} M^4 \mathrm{tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma \frac{1}{M^4} F_{\mu\nu}F_{\lambda\sigma}(x)\right]$

$$= \frac{-e^2}{32\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}(x)$$

and $\mathcal{J} = \exp\left[-i\int d^4x\, \alpha(x)\left(\frac{e^2}{32\pi^2}\cdot \epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}(x)\right)\right.$

Functional integral then becomes

original $\quad Z = \int \mathcal{D}\psi\,\mathcal{D}\bar\psi \exp\left[i\int d^4x\left(\bar\psi(i\not{D})\psi\right)\right]$

$$\psi'(x) = (1 + i\alpha(x)\gamma^5)\psi(x)$$

$$\bar\psi'(x) = \bar\psi(1 + i\alpha(x)\gamma^5)$$

$$\int d^4\,\bar\psi'(i\not{D})\psi' = \int d^4x\left(\bar\psi(i\not{D})\psi + \alpha(x)\,\partial_\mu(\bar\psi\gamma^\mu\gamma^5)\psi\right)$$

but combine with $\mathcal{J}$:

$$Z = \int \mathcal{D}\psi\,\mathcal{D}\bar\psi \exp\left[i\int d^4x\left(\bar\psi(i\not{D})\psi + \alpha(x)\cdot\left(\partial_\mu j^{\mu 5} + \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}\right)\right)\right]$$