Lecture 5.

11-16-18.

Pauli-Villars calculation + Fujikawa's method.

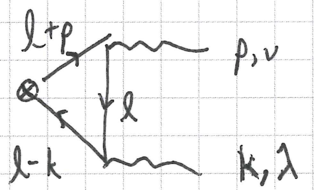
Calculate with Pauli-Villars regulator.

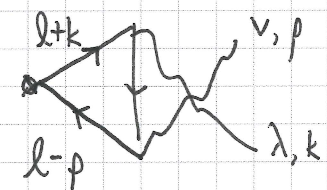
Introduce heavy fermion with mass M, opposite statistics. At end of calculation, take $M \to \infty$. Need one fermion for each light fermion.

$\Leftrightarrow$ For each integrand, subtract same expression with massive fermion propagator.

Note: massive regulator field lacks chiral symmetry.

Recall:





Adapt expression from before,

$$i\mathcal{M}_1^{\mu\nu\lambda} = (+1)\int \frac{d^4 l}{(2\pi)^4}\,\mathrm{Tr}\left[i\gamma^\mu\gamma^5 \frac{i(\not{l}-\not{k}+M)}{(l-k)^2-M^2+i\epsilon}\cdot(-ie\gamma^\lambda) \cdot \frac{i(\not{l}+M)}{l^2-M^2}(-ie\gamma^\nu)\frac{i(\not{l}+\not{p}+M)}{(l+p)^2-M^2}\right]$$

from only the regulator fermion.

$\mathcal{M}_2$ obtained from $(p,\nu) \longleftrightarrow (k,\lambda)$

Recall that using naive γ^5 in 4D led to complete cancellation of the contribution from light fermion. The only complication was that dim. reg. moved the loop integration to d-dim.

Earlier: $q_\mu \gamma^\mu\gamma^5 = (\not{l}+\not{p})\gamma^5 + \gamma^5(\not{l}-\not{k}) - 2\gamma^5 \not{l}_\perp$

Now, we can keep loop integral in 4D, so no subtlety with γ^5 and the massless fermion loop cancels itself (after shifting).

Again, evaluate divergence from regulator fermion.

$$i\mathcal{M}_1^{\nu\lambda} = iq_\mu \mathcal{M}_1^{\mu\nu\lambda} = -(+1)\int \frac{d^4\ell}{(2\pi)^4} e^2 \,\mathrm{Tr}\left[\not{q}\gamma^5 \frac{(\not{\ell}-\not{k}+M)}{(\ell-k)^2-M^2+i\epsilon}\gamma^\lambda \frac{(\not{\ell}+M)}{\ell^2-M^2+i\epsilon} \cdot \gamma^\nu \frac{(\not{\ell}+\not{p}+M)}{(\ell+p)^2-M^2+i\epsilon}\right]$$

Now, $\not{q}\gamma^5 = (\not{\ell}+\not{p}-\not{\ell}+\not{k}+2M)\gamma^5 - 2M\gamma^5$

$$= (\not{\ell}+\not{p}+M)\gamma^5 + \gamma^5[\not{\ell}-\not{k}+M] - 2M\gamma^5$$

As before, regulator fermion diagrams cancel from the first two terms after shifting & including the second diagram.

Remaining piece:

$$i\mathcal{M}_1^{\nu\lambda} = -e^2\int\frac{d^4\ell}{(2\pi)^4}\,\mathrm{Tr}\left[-2M\gamma^5 \frac{(\not{\ell}-\not{k}+M)}{(\ell-k)^2-M^2+i\epsilon}\gamma^\lambda \frac{(\not{\ell}+M)}{\ell^2-M^2+i\epsilon}\gamma^\nu \frac{(\not{\ell}+\not{p}+M)}{(\ell+p)^2-M^2+i\epsilon}\right]$$

Remember: Trace over odd # of γ vanishes.

Also, trace with γ^5 vanishes if 3 or fewer γ matrices.

$\Rightarrow$ Need 4 γ matrices in trace.

Since $\mathrm{Tr}[\alpha\beta\gamma\delta 5] = -4i\epsilon^{\alpha\beta\gamma\delta}$, we can drop terms that are symmetric in momenta.

Lastly, integration over ℓ forces even powers of ℓ.

Only surviving piece:

$$= -e^2\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell-k)^2-M^2+i\epsilon}\,\frac{1}{\ell^2-M^2+i\epsilon}\,\frac{1}{(\ell+p)^2-M^2+i\epsilon}$$

$$\cdot\,\mathrm{Tr}\left[-2M\gamma^5(-\not{k})\gamma^\lambda M\gamma^\nu\not{p}\right]$$

$$= -e^2(2M^2)(-4i\epsilon^{\alpha\lambda\nu\beta})\,k_\alpha p_\beta$$

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell-k)^2-M^2+i\epsilon}\;\frac{1}{\ell^2-M^2+i\epsilon}\;\frac{1}{(\ell+p)^2-M^2+i\epsilon}$$

A proper computation would involve Wick-rotating and explicit calculation as $M \to \infty$. For our purposes, it is sufficient to rescale $l = M q$ for the integration

$$\approx M^4 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(M^2 q^2 - M^2 + i\epsilon)} \frac{1}{(M^2 q^2 - M^2 + i\epsilon)} \frac{1}{(M^2 q^2 - M^2 + i\epsilon)}$$

$$= \frac{1}{M^2} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 - 1 + i\epsilon)^3}$$

Wick-rotation: $q_E^0 = -i q^0$

Peskin + Schroeder, p. 249

$$\int \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 + \Delta)^3} = \frac{1}{(4\pi)^2} \frac{\Gamma(1)}{\Gamma(3)} \frac{1}{\Delta}$$

from Wick-rotation

$$i\mathcal{M}_1^{\nu\lambda} = 8 i e^2 M^2 \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta \cdot \frac{1}{16\pi^2 M^2} \frac{1}{2} \cdot \frac{1}{(-1)} (-i)$$

$$i\mathcal{M}_1^{\nu\lambda} = \frac{-e^2}{4\pi^2} \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta = \frac{e^2}{4\pi^2} \epsilon^{\alpha\lambda\beta\nu} k_\alpha p_\beta$$

Again, $\mathcal{M}_2$ obtained from $(p,\nu) \leftrightarrow (k,\lambda)$ gives same

$$i\mathcal{M}_{tot} = \frac{e^2}{2\pi^2} \epsilon^{\alpha\lambda\beta\nu} k_\alpha p_\beta$$ as before using dim. reg.

Fujikawa's method

Main point: Fermionic measure in path integral is not invariant under axial rotation. Non-trivial Jacobian leads to modification of axial current conservation law.

P+S, (19.61) Originally, have $Z = \int \mathcal{D}\psi \mathcal{D}\bar\psi \exp\left[i \int d^4x\, \bar\psi (i\not{D}) \psi \right]$

functional measure.

After change of variables,

(19.79) $$Z = \int \mathcal{D}\psi \mathcal{D}\bar\psi \exp\left[i \int d^4x \left(\bar\psi (i\not{D})\psi + \alpha(x) \left\{ \partial_\mu j^{\mu 5} + \frac{e^2}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right\} \right) \right]$$

with α as the symmetry transformation parameter.