Lecture 3. Continue 4D chiral anomaly.

11-2-18

γ^5 prescription: $\{\gamma^\mu, \gamma^5\} = 0$ for $\mu = 0,1,2,3$

$[\gamma^\mu, \gamma^5] = 0$ otherwise

Then $\ell = \ell_{\parallel} + \ell_{\perp}$,

$$\not{\ell}\gamma^5 = -\gamma^5\not{\ell} + 2\gamma^5\not{\ell}_\perp$$

Divergence of vertex

$$i\mathcal{M}_1^{\nu\lambda} = e^2\int\frac{d^d\ell}{(2\pi)^d}\,\mathrm{Tr}\left[-2\gamma^5\not{\ell}_\perp\frac{(\not{\ell}-\not{k})}{(\ell-k)^2}\gamma^\lambda\frac{\not{\ell}}{\ell^2}\gamma^\nu\frac{(\not{\ell}+\not{p})}{(\ell+p)^2}\right]$$

Second diagram has $p,\nu \leftrightarrow k,\lambda$

Perform trace & loop integration.

Feynman params: $\dfrac{1}{D_1 D_2 D_3} = 2\displaystyle\int\frac{dx\,dy\,dz\;\delta(x+y+z-1)}{\left(z\ell^2 + x(\ell-k)^2 + y(\ell+p)^2\right)^3}$

Since $k^2 = p^2 = 0$ $\quad = 2\displaystyle\int\frac{dx\,dy}{(\ell^2 - 2x\,\ell\cdot k + 2y\,\ell\cdot p)^3}$

$$= e^2\int\frac{d^d\ell}{(2\pi)^d}\int dx\,dy\,\frac{\mathrm{Tr}[-4\gamma^5\not{\ell}_\perp(\not{\ell}-\not{k})\gamma^\lambda\not{\ell}\gamma^\nu(\not{\ell}+\not{p})]}{(\ell^2-2x\,\ell\cdot k+2y\,\ell\cdot p)^3}$$

Shift: $\ell - xk + yp \equiv L \Rightarrow L^2 = (\ell - xk + yp)^2 = \ell^2 - 2x\,\ell\cdot k + 2y\,\ell\cdot p + (-xk+yp)^2$

Denominator: $(L^2 - (-xk+yp)^2)^3$

$$\Rightarrow \Delta \equiv (-xk+yp)^2 = -2xy\,k\cdot p$$

$$= e^2\int\frac{d^d\ell}{(2\pi)^d}\int\frac{dx\,dy}{(L^2-\Delta)^3}\,\mathrm{Tr}[-4\gamma^5\not{\ell}_\perp(\not{\ell}-\not{k})\gamma^\lambda\not{\ell}\gamma^\nu(\not{\ell}+\not{p})]$$

Let $P \equiv -xk+yp$, then trace is

$$\mathrm{Tr}[-4\gamma^5\not{L}_\perp(\not{L}-\not{P}-\not{k})\gamma^\lambda(\not{L}-\not{P})\gamma^\nu(\not{L}-\not{P}+\not{p})]$$

1. Trace nonzero iff even # of γ matrices. Also, γ^5 with two γ matrices vanishes.

2. Loop integration vanishes for odd ℓ.

Leading power in L:

$$\mathrm{Tr}[-4\gamma^5 \not{\ell}_\perp \not{L} \gamma^\lambda \not{L} \gamma^\nu \not{L}]$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \not{L}\gamma^\nu = 2L^\nu - \gamma^\nu \not{L}$$

$$= \mathrm{Tr}[-4\gamma^5 \not{\ell}_\perp \not{L} \gamma^\lambda (2L^\nu) \not{L}] - \mathrm{Tr}[-4\gamma^5 \not{\ell}_\perp \not{L} \gamma^\lambda \gamma^\nu] L^2$$

First term vanishes,

Recall $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] = -4i\epsilon^{\mu\nu\rho\sigma}$

$$\mathrm{Tr}[\gamma^5 \not{\ell}_\perp \not{L} \gamma^\lambda \not{L}](-8L^\nu)$$

$$\propto (-8L^\nu)(-4i\epsilon^{\mu\alpha\lambda\beta} \ell_{\perp\mu} L_\alpha L_\beta),$$ which is symmetric in $\alpha\beta$.

Second term vanishes:

$$+4L^2\,\mathrm{Tr}[\gamma^5 \not{\ell}_\perp \not{L} \gamma^\lambda \gamma^\nu] = 0$$

since we need to have $\not{L} \to \not{\ell}_\perp$ in order to have the loop integral be even in $\ell_\perp$, but then the trace only has $\mathrm{Tr}[\gamma^5\gamma^\lambda\gamma^\nu] = 0$.

Two powers of L:

$$\mathrm{Tr}[-4\gamma^5 \not{\ell}_\perp \not{L} \gamma^\lambda (-\not{P}) \gamma^\nu (-\not{P} + \not{p})]$$
$$+ \mathrm{Tr}[-4\gamma^5 \not{\ell}_\perp (-\not{P} - \not{k}) \gamma^\lambda \not{L} \gamma^\nu (-\not{P} + \not{p})]$$
$$+ \mathrm{Tr}[-4\gamma^5 \not{\ell}_\perp (-\not{P} - \not{k}) \gamma^\lambda (-\not{P}) \gamma^\nu \not{L}]$$

All $\not{L} \to \not{\ell}_\perp$ to get nonzero loop integral.

$\not{\ell}_\perp$ anticommute with other Dirac matrices, $\{\not{\ell}_\perp, \gamma^\mu\} = 0$

as long as other Dirac matrices live in 4D. (cf. $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$)

$$= \mathrm{Tr}[-4\ell_\perp^2 \gamma^5 \gamma^\lambda (-\not{P}) \gamma^\nu \not{p}]$$
$$+ \mathrm{Tr}[-4\ell_\perp^2 \gamma^5 (-\not{P} - \not{k}) \gamma^\lambda \gamma^\nu (-\not{P} + \not{p})]$$
$$+ \mathrm{Tr}[-4\ell_\perp^2 \gamma^5 (-\not{k}) \gamma^\lambda (-\not{P}) \gamma^\nu]$$

First use P-dependent terms in second trace

$$= -4\ell_\perp^2 \Big(\mathrm{Tr}[\gamma^5 \gamma^\lambda x\not{k} \gamma^\nu \not{p}] + \mathrm{Tr}[\gamma^5 (-\not{P}) \gamma^\lambda \gamma^\nu \not{p}]$$
$$+ \mathrm{Tr}[\gamma^5 (-\not{k}) \gamma^\lambda \gamma^\nu (-\not{P})] + \mathrm{Tr}[\gamma^5 (-\not{k}) \gamma^\lambda (-y\not{p}) \gamma^\nu] \Big)$$

$$= -4\ell_\perp^2 \Big(\mathrm{Tr}[\gamma^5 \gamma^\lambda x\not{k} \gamma^\nu \not{p}] + \mathrm{Tr}[\gamma^5 x\not{k} \gamma^\lambda \gamma^\nu \not{p}]$$
$$+ \mathrm{Tr}[\gamma^5 \not{k} \gamma^\lambda \gamma^\nu y\not{p}] + \mathrm{Tr}[\gamma^5 \not{k} \gamma^\lambda y\not{p} \gamma^\nu] \Big)$$

$$= -4\, l_\perp^2 \,(-4i)\,(k_\alpha p_\beta)\big(x\,\epsilon^{\lambda\alpha\nu\beta} + x\,\epsilon^{\alpha\lambda\nu\beta} + y\,\epsilon^{\alpha\lambda\nu\beta} + y\,\epsilon^{\alpha\lambda\beta\nu}\big)$$

$$= 16 i\, l_\perp^2\, k_\alpha p_\beta\,(0) \boxed{=0}$$

So P-dependent terms cancel.

Remaining term:

$$\mathrm{Tr}\left[-4\, l_\perp^2\, \gamma^5 (-\not{k})\, \gamma^\lambda \gamma^\nu \not{p}\right]$$

$$= 4\, l_\perp^2 (-4i)\, \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta$$

$$= -16\, l_\perp^2\, i\, \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta$$

If no powers of L in trace, then integral over $l_\perp$ vanishes.

So
$$i\mathcal{M}_1^{\nu\lambda} = \frac{e^2}{(2\pi)^d}\int d^d l \int \frac{dx\,dy}{(L^2-\Delta)^3}(-16)\, i\, l_\perp^2\, \epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta$$

Use $l_\perp^2 = \frac{(d-4)}{d} L^2$ + $\displaystyle\int \frac{d^d L}{(2\pi)^d}\frac{L^2}{(L^2-\Delta)^3} = \frac{(-1)^2}{(4\pi)^{d/2}}\frac{id}{2} \times \frac{\Gamma(2-\frac{d}{2})}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$

$$i\mathcal{M}_1^{\nu\lambda} = e^2 \int dx\,dy\, \frac{(-16i)}{16\pi^2}\epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta\, \frac{(d-4)}{d}\cdot\frac{id}{2}\,\frac{1}{2}\cdot\left(\frac{2}{\epsilon} - \log \Delta - \gamma + \log 4\pi\right)$$

for $d = 4-\epsilon$.

Note $d=4-\epsilon$ gives $\frac{d-4}{\epsilon}$, which is ill-defined unless the divergence shows up exactly as a $\frac{1}{\epsilon}$ pole. If we naively used $d=4$, then $l_\perp^2 \to \frac{(d-4)}{d} l^2 = 0$.

We get a constant when $\epsilon \to 0$, remaining terms in $(\dots)$ vanish.

$$i\mathcal{M}_1^{\nu\lambda} = e^2 \int dx\,dy\, \frac{1}{\pi^2}\epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta \frac{-1}{4}(2), \text{ since } d-4 = -\epsilon$$

$$= -\frac{e^2}{4\pi^2}\epsilon^{\alpha\lambda\nu\beta} k_\alpha p_\beta = +\frac{e^2}{4\pi^2}\epsilon^{\alpha\lambda\beta\nu} k_\alpha p_\beta, \text{ since } \int dx\,dy = \frac{1}{2}$$

(4)

For $\mathcal{M}_2$, swap $p,\nu \leftrightarrow k,\lambda$

$$i\mathcal{M}_2^{\nu\lambda} = \frac{e^2}{4\pi^2}\,\epsilon^{\alpha\nu\beta\lambda}\, p_\alpha k_\beta = \frac{e^2}{4\pi^2}\,\epsilon^{\beta\nu\alpha\lambda}\, p_\beta k_\alpha \quad \text{(relabel)}$$

$$= \frac{e^2}{4\pi^2}\,\epsilon^{\alpha\lambda\beta\nu}\, k_\alpha p_\beta = i\mathcal{M}_1^{\nu\lambda}$$

So both diagrams add up.

$$i\mathcal{M}_{tot}^{\nu\lambda} = \frac{e^2}{2\pi^2}\,\epsilon^{\alpha\lambda\beta\nu}\, k_\alpha p_\beta = \langle \partial_\mu j^{\mu 5} \rangle$$

Associate with Feynman rule

$$\langle \partial j^{\mu 5} \rangle = \langle -\frac{e^2}{16\pi^2}\,\epsilon^{\alpha\nu\beta\lambda}\, F_{\alpha\nu} F_{\beta\lambda} \rangle$$

Operator equivalence (not just expectation value) is demonstrated via fujikawa's method.