Lecture 2.
October 26, 2018

Last time: Motivation for Advanced QFT.
- SM as a QFT, no input parameters from exp.

Schwinger model

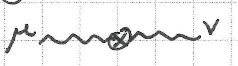
Found $i\Pi_2^{\mu\nu}(q) = \frac{ie^2}{\pi}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)$ using dim. reg.

Dim. reg. guarantees $g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}$ structure:

satisfies QED Ward identity when contracting with q^μ or q^ν.

This is not a given. Coefficient of $g^{\mu\nu}$ + $q^\mu q^\nu$ are driven by leading UV divergence + finite in 2D.

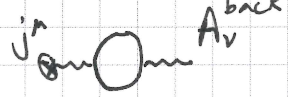
Can adjust $g^{\mu\nu}$ coefficient via counterterm $\delta\mathcal{L} = \delta a\, A_\mu A^\mu$

$\Rightarrow$ 

$= i\delta a\, g^{\mu\nu}$

Leads to violation of Ward identity + gauge invariance.

Check $j^{\mu 5}$ conservation.

First, $\langle j^\mu\rangle$ from 

$$\langle j^\mu(q)\rangle = \frac{i}{e}\left(i\,\Pi^{\mu\nu}(q)\right)A_\nu(q) = -\frac{e}{\pi}A_\nu^{back}\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)$$

Satisfies current conservation $q_\mu\langle j^\mu(q)\rangle = 0$ for arbitrary background field. A_ν.

Then, $\langle j^{\mu5}\rangle = -\epsilon^{\mu\nu}\langle j_\nu\rangle$ from $\gamma^\mu\gamma^5 = -\epsilon^{\mu\nu}\gamma_\nu$

$$= \epsilon^{\mu\nu}\frac{e}{\pi}A_\alpha^{back}\left(g_\nu^{\ \alpha}-\frac{q_\nu q^\alpha}{q^2}\right)$$

$$= \epsilon^{\mu\nu}\frac{e}{\pi}\left(A_\nu^{back} - A\cdot q\,\frac{q_\nu}{q^2}\right)$$

$$q_\mu\langle j^{\mu5}\rangle = \epsilon^{\mu\nu}\frac{e}{\pi}q_\mu A_\nu^{back} - \epsilon^{\mu\nu}\frac{e}{\pi}A\cdot q\,\cancel{\frac{q_\mu q_\nu}{q^2}} \to 0$$ by anti-symmetry

Fourier transform

$$\partial_\mu j^{\mu5} = \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$$ 2D axial vector anomaly equation

* The classical axial current is not conserved. ②

How to understand? What is the physical consequence?

Also, what about photon mass term?

Can derive axial anomaly from second calculation.

P&S, 19.1, p.655.

Shifman, 33.3, p.305. Schwinger (aka ϵ-) splitting.

To understand $\partial_\mu j^{\mu 5} = \frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu} = \frac{e}{2\pi} \cdot 2\partial_\mu (\epsilon^{\mu\nu} A_\nu)$

Consider Dirac sea, spectrum of fermion energy eigenstates.

$\epsilon^{\mu\nu} F_{\mu\nu}$ is a total derivative.

$$\int d^2x\ \partial_\mu j^{\mu 5} = N_R - N_L \Rightarrow \int d^2x\ \frac{e}{\pi} \partial_\mu (\epsilon^{\mu\nu} A_\nu) = N_R - N_L$$

See that behavior of gauge field at spatial boundary determines non-conservation.

As an ~~exp~~ example, take spatially const. A' and slowly varying

Take system to have length L, periodic bdry conditions

Note Wilson line is nontrivial $e^{-ie\int_0^L dx' A'}$

so const. A' cannot be removed by gauge transformation.

Hamiltonian $\left(\text{cf. } H = \int dx' \left[\pi(x) \dot\phi(x) - \mathcal{L} \right] \right)$

$$H = \int dx\ \psi^\dagger (-i\alpha' D_1)\psi$$

or $$H = \int dx \left\{ -i\psi_+^\dagger (\partial_1 - ieA')\psi_+ + i\psi_-^\dagger (\partial_1 - ieA')\psi_- \right\}$$

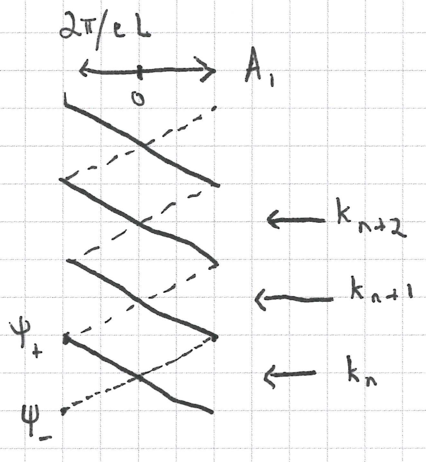
Eigenstates are $e^{ik_n x}$ with $k_n = \frac{2\pi n}{L}$, $n = -\infty \dots \infty$.

Energies are $\psi_+ : \quad E_n = +(k_n - eA')$

$\psi_- : \quad E_n = -(k_n - eA')$

As usual, fill negative energy levels and interpret holes created ~~above~~ among filled states as antiparticles.

Spectrum:



At $A_1 = 0$, degenerate for ψ_+, ψ_-.

As A_1 increases from $-\frac{\pi}{eL}$ to $\frac{\pi}{eL}$, ψ_+ falls and ψ_- rises.

For $\Delta A_1 = \frac{2\pi}{eL}$, the shift in spectrum is unchanged.

Fermion levels are restructured $\Rightarrow$ essence of chiral anomaly.

As $\Delta A_1 = \frac{2\pi}{eL}$, vacuum state now has one LH / ψ_- particle and one RH / ψ_+ hole.

Quantum #s of fermion sea: net charge is unchanged + gauge invariance is maintained. But

$$\int d^2x \left(\frac{e}{\pi}\epsilon^{\mu\nu}F_{\mu\nu}\right) = \int dt\, dx\, \frac{e}{\pi}\partial_0 A_1 = \frac{e}{\cancel{\frac{1}{2}}\pi} L(-\Delta A') = -2 = N_R - N_L$$

The non-conservation of $j^{\mu 5}$ requires that $j^{\mu 5}$ is not a good quantum number of vacuum. $N_R + N_L = j^\mu$ is conserved, $N_R - N_L = j^{\mu 5}$ is not.

Critically relies on ∞ # of levels. + Also topological connection from behavior of gauge fields at ∞.

Can interpret as ground state / vacuum state appearance of particle + hole. Can also interpret as UV ~~dis~~appearance of particle beyond UV cutoff + one hole appears in UV.

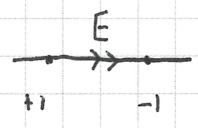
Shifman, 33.3. More convenient in non-Abelian with confinement.

What about photon mass?

Will see in second half when discuss instantons + Θ-vacuum.

Instantons lead to chiral condensate of $\psi + \bar{\psi}$, meson has mass $\frac{e^2}{\pi}$ + is non-interacting for $m=0$.

Confinement occurs for $\psi + \bar{\psi}$:



E field for two charges is constant, potential increases linearly.

Asymptotic states are <u>not</u> those elementary fields in Lagrangian.

Now consider 4D QED with chiral coupling.

Exhibit three calculations: dim. reg. w/ γ^5 prescription, Pauli-Villars, Fujikawa's method.

In words: Many ways to regulate divergences in QFT.

Green, Schwarz, Witten, Superstring Theory Vol II

Particularly, Feynman diagrams with classically conserved currents can be divergent.

Anomalies arise when these diagrams do not admit a regulator compatible with simultaneous conservation of all attached currents.

Gauge anomalies, nesessarily for chiral gauge symmetry, cannot be removed by counterterms.

$$\mathcal{L} = \bar{\Psi}\,(i\not{\partial} - m)\,\psi \;-\; \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Again, for $m=0$, classically get $\partial_\mu j^{\mu 5} = 0$, $j^{\mu 5} = \bar{\Psi}\gamma^\mu\gamma^5\psi$.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \qquad \sigma^\mu = (1, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

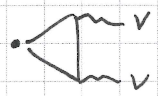
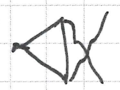
$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\mathcal{L}$ is again separable for $m=0$:

$$\mathcal{L} = \psi_R^\dagger \, i\sigma\cdot D\, \psi_R + \psi_L^\dagger \, i\bar{\sigma}\cdot D\,\psi_L - \tfrac{1}{4} f_{\mu\nu}F^{\mu\nu}$$

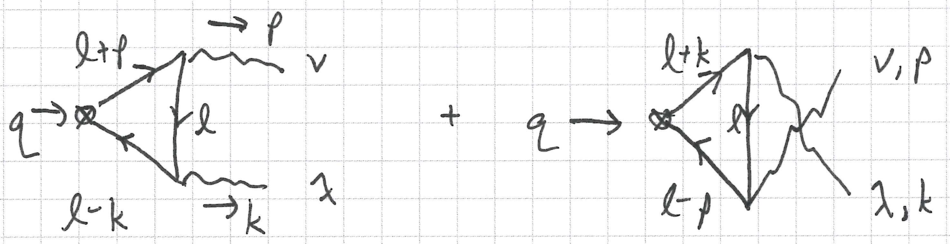
$\psi_R \sim h = \frac{1}{2}$

$\psi_L \sim h = -\frac{1}{2}$.

Study $\langle \partial_\mu j^{\mu 5}\rangle = i q_\mu \Big[A^{\mu 5}$  $+ A^{\mu 5}$  $\Big]$

from $\int d^4x\, e^{-iq\cdot x} \langle \partial_\mu j^{\mu 5}\rangle = \int d^4x\, e^{-iq\cdot x}\langle i q_\mu j^{\mu 5}\rangle$

Same momentum convention from P+S.



Full vertex is difficult + ambiguous (Adler method.)

$$iM^{\mu\nu\lambda} = (-1)\int \frac{d^4l}{(2\pi)^4} \mathrm{Tr}\Big[(i\gamma^\mu\gamma^5)\frac{i(\not{l}-\not{k})}{(l-k)^2}(-ie\gamma^\lambda)\frac{i\not{l}}{l^2}(-ie\gamma^\nu)\frac{i(\not{l}+\not{p})}{(l+p)^2}\Big]$$

$$+ (p,\nu) \longleftrightarrow (k,\lambda)$$

Dot with q_μ:

Naively, $\not{q}\gamma^5 \equiv (\not{l}+\not{p}-\not{l}+\not{k})\gamma^5 \overset{?}{=} (\not{l}+\not{p})\gamma^5 + \gamma^5(\not{l}-\not{k})$

Note: $\not{a}\not{a} = a_\mu a_\nu \gamma^\mu\gamma^\nu = \frac{1}{2} a_\mu a_\nu (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = a^2$

First diagram piece

$$iM^{\nu\lambda} = i q_\mu M^{\mu\nu\lambda} = +e^2 \int \frac{d^4 l}{(2\pi)^4}\mathrm{Tr}\Big[\gamma^5\gamma^\lambda \frac{\not{l}}{l^2}\gamma^\nu\frac{(\not{l}+\not{p})}{(l+p)^2} + \gamma^5\frac{(\not{l}-\not{k})}{(l-k)^2}\gamma^\lambda\frac{\not{l}}{l^2}\gamma^\nu\Big]$$

$$= e^2 \int \frac{d^4 l}{(2\pi)^4} \, \mathrm{Tr}\left[-\gamma^5 \frac{\not{l}}{l^2} \gamma^\nu \frac{(\not{l}+\not{p})}{(l+p)^2} \gamma^\lambda + \gamma^5 \frac{(\not{l}-\not{k})}{(l-k)^2} \gamma^\lambda \frac{\not{l}}{l^2} \gamma^\nu \right]$$

Shift $l \to l+k$ in second term.

$$= e^2 \int \frac{d^4 l}{(2\pi)^4} \, \mathrm{Tr}\left[-\gamma^5 \frac{\not{l}}{l^2} \gamma^\nu \frac{(\not{l}+\not{p})}{(l+p)^2} \gamma^\lambda + \gamma^5 \frac{(\not{l})}{l^2} \gamma^\lambda \frac{(\not{l}+\not{k})}{(l+k)^2} \gamma^\nu \right]$$

Cancels $(p,\nu) \leftrightarrow (k,\lambda)$

But need to be careful about γ^5 in dim. reg.

Recall dim. reg. preserves gauge invariance of vector current.

Adopt 't Hooft Veltman prescription. Nucl. Phys. B 44, 189 (1972).

γ^5 anti-commutes for $\mu = 0,1,2,3$, commutes otherwise.

Loop integral lives in d-dim. $l = l_\parallel + l_\perp$

$$q_\mu \gamma^\mu \gamma^5 = (\not{l}+\not{k})\gamma^5 + \gamma^5(\not{l}-\not{p}) - 2\gamma^5 \not{l}_\perp \quad \Leftarrow$$ full l momentum then correction.

Above calculation gets additional term.

$$i\mathcal{M}^{\nu\lambda} = e^2 \int \frac{d^4 l}{(2\pi)^4} \, \mathrm{Tr}\left[-2\gamma^5 \not{l}_\perp \frac{(\not{l}-\not{k})}{(l-k)^2} \gamma^\lambda \frac{\not{l}}{l^2} \gamma^\nu \frac{(\not{l}+\not{p})}{(l+p)^2} \right]$$

Lecture 2. Supplement.

Global vs. gauge symmetries.

$U(N)$	$SU(N) \times U(1)$
$= SU(N) \times U(1)$	Semi-simple Lie groups
$\Downarrow$	
study from only free theory Lagrangian	

Same symmetry groups, difference is whether global or local transformation.

Use to consider full symmetry structure + gauging particular subgroup.

Chiral global vs. gauge:

Consider SM: Chiral symmetries are $SU(2)_L \times U(1)_Y$. QCD is non-chiral

$Q_L \sim (3, 2, 1/6)$ $\quad u_R \sim (3, 1, 2/3)$ $\quad d_R \sim (3, 1, -1/3)$

$L_L \sim (1, 2, -1/2)$ $\quad e_R \sim (1, 1, -1)$

Consider pions + up + down quarks.

$$\mathcal{L} = \bar{u}\, i \not{D}\, u + \bar{d}\, i \not{D}\, d - m_u \bar{u} u - m_d \bar{d} d$$

$\not{D}$ includes color interaction. No projection P_L or P_R in gauge interactions, in contrast to unbroken EW symmetry.

Study π-spectrum. ~~This~~ Special limit: If $m_u, m_d \ll \Lambda_{QCD}$ and are negligible in π masses, then $\mathcal{L}$ has extra global $U_u(2) \times U_d(2)$ symmetry. $U_u(2) \times U_d(2) = U_L(2) \times U_R(2)$ for $q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}$ + $q_R \equiv \begin{pmatrix} u_R \\ d_R \end{pmatrix}$, and then $U_L(2) \times U_R(2) \to U_V(2) \times U_A(2) = SU_V(2) \times U(1)_V \times SU_A(2) \times U(1)_A$.

The $U(1)_A$ is anomalous, $SU(2)_A$ is explicitly broken if turn on $m_u = m_d \neq 0$, and both broken by EM.