Lecture 15.
Extra.

22.2.19.

Violation of Cluster Decomposition.

cf. Shifman, 18.2, 33.6.

We cannot "turn off" instantons and only keep a classical, perturbative vacuum $|n\rangle$ as the quantum vacuum instead of the true Θ vacuum, $|\theta\rangle = \sum_{n \in \mathbb{Z}} e^{in\theta} |n\rangle$.

One reason is the full invariance under gauge transformations includes "large" gauge transformations that change $|n\rangle$.

A separate reason is that such a choice would violate cluster decomposition:

For two operators $O_1(x_1)$ & $O_2(x_2)$, the vev of the T product: $\langle T\{O_1\, O_2\}\rangle \to \langle O_1\rangle\langle O_2\rangle$ as $|x_1 - x_2| \to \infty$.

Consider $A(t) = \langle n|\, T\{ O^+(t), O(0)\}\,|n\rangle$

$$O(t) = \int \bar{\psi}(t,x)\,(1+\gamma^5)\,\psi(t,x)\,dx$$

O changes axial charge of the state by 2 units & O^+ changes it back, so $A(t) \neq 0$. As $t \to \infty$, then $A(t) \to$ const, so operators $\bar{\psi}(1 \pm \gamma^5)\psi$ acquire a nonzero vev. But for $|n\rangle$ vacuum, then $\langle \bar{\psi}(1\pm\gamma^5)\psi\rangle = 0$, since the fermion & corresponding hole in Dirac sea is orthogonal to $|n\rangle$, which has unchanged fermion numbers.

Instantons + fermions and Θ-vacua + the Strong CP problem

Recalculate, in $A_0=0$ gauge, transition from $|n\rangle$ to $|n+k\rangle$ with fermions.

$$\lim_{\tau\to\infty} \langle n+k| e^{-H\tau} |n\rangle$$
$$= \int (\mathcal{D}A_\mu)_k \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \; e^{-S_E}$$

$$S_E = \int d^4x \left(\frac{1}{4g^2} G^a_{\mu\nu} G^{a\mu\nu} + i\bar{\Psi}(\not{D} - \not{m}^{\,0})\psi \right)$$

In path integral, ψ + $\bar{\Psi}$ are totally independent:

cf. Coleman, Uses of Instantons

$$\psi(x) = \psi_0(x) + c_n \psi_n(x)$$
$$\bar{\Psi}(x) = \bar{\Psi}_0(x) + \bar{c}_n \bar{\Psi}_n(x)$$

$\psi_0, \bar{\Psi}_0$ classical solutions to Euclidean EOM.

$\psi_n, \bar{\Psi}_n$ are complete sets of orthonormal, commuting functions. $c_n, \bar{c}_n$ are anti-commuting coeff.

$$c^2 = \bar{c}^2 = 0 \qquad f(c,\bar{c}) = a_0 + a_1 c + \bar{a}_1 \bar{c} + a_2 c\bar{c}$$

Rules: $\int dc\, f(c+\epsilon) = \int dc\, f(c)$ translation invariance.

$$\int dc\; 1 = \int d\bar{c} = 0 \qquad \int dc\; c = 1 \qquad \int d\bar{c}\; \bar{c} = 1$$

$$\int dc \int d\bar{c}\; \bar{c}c = 1$$

$$1 - c\bar{c} = \exp(1 - c\bar{c}) \;\; + \;\; \int dc\, d\bar{c}\, \exp(1 - c\bar{c}) = 1.$$

Define $\int \mathcal{D}\psi^\dagger \mathcal{D}\psi = \int \prod_j (i\, d\bar{c}_j\, dc_j)$

$$\int d^4x\, \psi_a^\dagger \psi_b = \int d^4x\, \bar{\Psi}_a \bar{\Psi}_b^\dagger = \delta_{ab}, \text{ orthonormal.}$$

Calculate $\int \mathcal{D}\psi^\dagger \mathcal{D}\psi\, \exp\left(i \int d^4x\, \psi^\dagger \cdot A\, \psi\right)$

$$A\psi_i = A_{ij}\psi_j.$$

$$\exp\left(i\int d^4x\ \psi^+ A\psi\right) = \exp\left(i\bar{c}_j A_{jk} c_k\right)$$

$$\Rightarrow \int D\psi^+ D\psi \exp\left(i\int d^4x\ \psi^+ A\psi\right)$$

$$= \int \prod_j \left(i\, d\bar{c}_j\, dc_j\right) \exp\left(i\bar{c}_j A_{jk} c_k\right)$$

$$= \text{Det } A.$$

Now, $\lim_{\tau\to\infty} \langle n+k | e^{-H\tau} | n\rangle$

$$= \int [DA_\mu]_k \int D\psi^+ D\psi\, e^{-S_E} = \text{const.} \int [DA_\mu]_k\, e^{-S_A}\, \text{Det}[M(A_\mu)]$$

where $M(A_\mu) \equiv i(\partial_\mu + A_\mu)\gamma_4\gamma_\mu$ in Euclidean.

Dirac operator in presence of external field A_μ.

Key point: for A_μ with non-trivial winding, then we saw the L-moving + R-moving fermions #s were not conserved. Then, under the operation of $(A_\mu)_{k=1}$, one LH eigenvalue must pass through 0 + one RH eigenvalue must pass through 0. So, functional integral is 0.

Spectral flow: fermion energy levels as gauge field flows along tunneling path.

Index theorem: Atiyah-Singer index theorem:

$$I(D) = k_+ - k_- = 2T(r)k = k.$$

Vacuum tunneling suppressed by massless fermions.

As a result, energy level splitting does not occur.

Cluster decomposition $\Rightarrow$ still have θ vacua.

All θ vacua have same E_0.

Different θ vacua obtained by rotation of massless fields into each other.

Anomaly: $\partial_\mu j^{\mu 5} = N_f \frac{g^2}{8\pi^2} \, tr \, F_{\mu\nu} \tilde{F}^{\mu\nu}$ → anomaly

$\partial_\mu j^\mu_A = \frac{g^2}{16\pi^2} \, tr \, F_{\mu\nu} \tilde{F}^{\mu\nu}$ → winding

$J^\mu_5 = j^\mu_5 - 2N_f \, j^\mu_A$ divergenceless

But j^μ_A = C-S current is gauge dependent.

Clear that chiral rotation absorbs winding #.

Calculate effective instanton interaction.

Instanton annihilates RH fermion of each flavor. + creates one LH fermion of each flavor.

Anti-instanton does opposite.

$$\epsilon^{ij\ldots l} \, \psi^\dagger_{i\alpha R} \, \psi^\dagger_{j\beta R} \cdots \psi^\dagger_{l\eta R} \, \psi_{m\lambda L} \psi_{n\kappa L} \cdots \psi_{p\sigma L} \, \epsilon^{mn\ldots p}$$

(i ↑ flavor, α ↑ spin)

't Hooft effective interaction.

$\det(\psi^\dagger_{iR} \psi_{jL})$ → manifestly violates $U(1)$ chiral symmetry.

Resolves η' problem.

Strong CP:

$$\mathcal{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 - \frac{1}{4}(W^i_{\mu\nu})^2 - \frac{1}{4}(F_{\mu\nu})^2$$

$$- \left(\lambda_d^{ij} \, Q^{\dagger i}_a \phi_a d^j_R + \lambda_u^{ij} \, Q^{\dagger i}_a \epsilon^{ab} \phi^*_b u^j_R \right) + h.c.$$

$$\lambda = \tilde{U}^\dagger \, \text{Diag} \, \tilde{V} = U^\dagger \, \text{Diag} \, V \, e^{i\phi}$$

($\tilde{U}$, $\tilde{V}$: $U(n)$; U, V: $SU(n)$)

$$\mathcal{L} = -\frac{1}{\sqrt{2}}(v+h)\left(\lambda^{ij}_d \, d^\dagger_L d_R + \lambda^{ij}_u \, u^\dagger_L u_R + h.c. \right)$$

$d^m_R = e^{i\phi} V_d \, d_R \qquad u^m_R = e^{i\phi} V_u \, u_R$

$d^m_L = U_d \, d_L \qquad u^m_L = U_u \, u_L$

$$\mathcal{L} = -[\, \bar{d}_L^{i} \, m_i^d \, d_R^i + \bar{u}_L^{i} \, m_i^u \, u_R + h.c.] \left(1 + \frac{h}{v}\right)$$

$$\delta\mathcal{L} = \frac{\phi \, n_f}{32\pi^2} (F\tilde{F})$$

For $\mathcal{L} = \frac{\theta}{32\pi^2} F\tilde{F}$, get $\bar{\theta} = \theta + \phi \cdot N_f$

$$= \theta + \arg\det M$$

$\bar{\theta} < 3 \cdot 10^{10}$ from neutron EDM.

Puzzle: Phase in CKM matrix is $\delta \sim 10^{-1}$ to 0.5.

Can easily arise in λ_u, λ_d with CPV phases of $\mathcal{O}(1)$.

But the combination in $\bar{\theta}$ is restricted to be $\mathcal{O}(10^{-10})$.

Solutions:

1. $m_u = 0$.
2. Nelson-Barr. Mass matrix ~~of~~ of quarks has CP symmetry, so arg vanishes, but quark submatrix has ~~vanish~~ nonzero CPV for weak interactions.
3. Peccei-Quinn sym. + axion.

   Axion couples to topological susceptibility of QCD + feels potential from θ- vacuum energy + dynamically minimizes energy to 0.