Lecture 14.

15.2.19

cf. Shifman, Chap. 18

Continue instantons + Θ vacuum.

Explicit form of $U_1(\vec{x}) = + \exp\left(i\pi \frac{\vec{x}\cdot\vec{\tau}}{(\vec{x}^2+\rho^2)^{1/2}} \right)$

Has winding number 1.

Can generalize to winding number n by $U_n = (U_1)^n$.

Recall $A_i(\vec{x})\big|_{vac} = i\, U_n(\vec{x})\, \partial_i U_n^\dagger(\vec{x})$ is pure gauge

+ corresponds to zero energy.

Matrices U_n, n are in different classes + cannot be continuously deformable into each other because of "holes" in the space of fields, with noncontractible loops around such "holes."

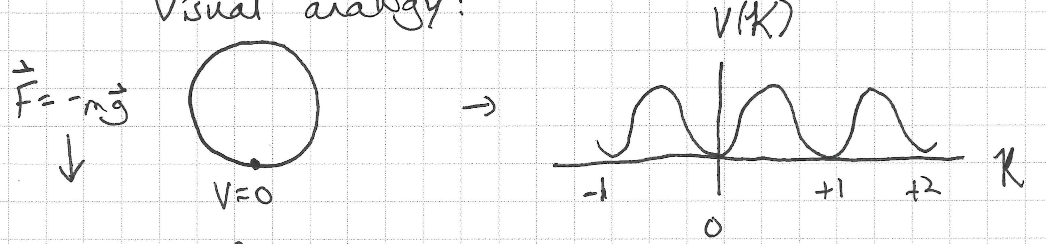
Consider degree of freedom corresponding to this winding number: Chern-Simons current.

$$K^\mu = 2\epsilon^{\mu\nu\alpha\beta}\left(A_\nu^a \partial_\alpha A_\beta^a + \frac{g}{3} f^{abc} A_\nu^a A_\alpha^b A_\beta^c \right)$$

$$\mathcal{K} = \frac{g^2}{32\pi^2} \int d^3x\, K_0(x)$$

$$\mathcal{K} = \text{winding } \# = n.$$

Visual analogy:



To formulate ground state, require periodic BC.

$$\Psi(\mathcal{K}+1) = e^{i\theta}\, \Psi(\mathcal{K})$$

$$\Psi_\theta = \sum_{n\in\mathbb{Z}} e^{in\theta}\, \Psi_n \quad \Rightarrow \text{ Superselection } \langle \Psi_{\theta'} | \mathcal{O} | \Psi_\theta \rangle = 0$$

for $\theta' \neq \theta$!

Instanton: field configuration $A_\mu(t,\vec{x})$ continuously interpolating between given $K=n$ & $K=n+1$ from $t=-\infty$ to $t=+\infty$ in Euclidean time. "BPST" = Belavin, Polyakov, Schwarz, Tyupkin PL 59B, 85 (1975).

Lagrangian formalism:

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} + \theta \frac{g^2}{32\pi^2} G^{a,\mu\nu} \tilde{G}^a_{\mu\nu}$$

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G^{a,\alpha\beta}$$

Comments: (1) leads to P & T violation. $n_{EDM} \Rightarrow \bar{\theta} < 10^{-10}$

(2) $G\tilde{G} = \partial_\mu K_\mu$, total derivative, but K_μ is not gauge invariant! Integrating over instanton field does not vanish. Cannot consider K_μ as physical observable, no restriction that this type of total derivative vanishes.

Moreover, since K^μ is not a physical current, it can have stronger singularities at spatial infinity.

(3) Superselection: amplitudes (inner products between wavefcns.) between states that belong to different θ vacua vanish. So in pure Y-M, no chiral fermions, we cannot transition to a different θ vacuum.

(4) For other SU(N) gauge groups, any continuous mapping of S^3 into G, with G a simple Lie group, can be continuously deformed into a mapping into an SU(2) subgroup of G.

Given $A_0 = 0$ gauge & the explicit form of the instanton with

$$U_1(\vec{x}) = +\exp\left(i\pi \frac{\vec{x}\cdot\vec{\tau}}{(\vec{x}^2+\rho^2)^{1/2}} \right),$$

have

$$\frac{\partial U_1(\vec{x})}{\partial |\vec{x}|} = \frac{(\rho\vec{\tau} + i\vec{x})^2}{|\vec{x}|^2 + \rho^2},$$

can calculate for $A_i(\vec{x})|_{vac} = i U_1(\vec{x})\, \partial_i U_1^\dagger(\vec{x})$ that

$$\int d^4x \, \mathrm{Tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{16\pi^2}{g^2}.$$

(Clearly, instanton contributions are non-perturbate (not seen as power expansion in g.))

For $A_0 = 0$ gauge, classical n-vacua labled by $|n\rangle$. Not eigenstates of Hamiltonian because of tunneling.

Recall QM of periodic potentials.

Consider 1+1 dim., $V(q) = V(q + 2\pi)$, $q \in [-\infty, \infty]$.

Let minima of $V = 0$ for $q = 2\pi N$, example $V = 1 - \cos q$.

i.) Expand around $q = 0$.

$$V(q) = \frac{1}{2} \omega^2 q^2 + \sum_{r=3}^{\infty} \lambda_r \frac{q^r}{r!}$$

Weak coupling condition $\lambda_r \ll \omega^2$.

Tunneling of low energy wavefcns. from one well to next is small.

If we neglect tunneling, can construct ground state wavefcn.

$u_0(q)$ in well around $q = 0$ with $E_0 = \frac{1}{2}\hbar\omega$, to lowest order in λ_r. There are infinitely many degenerate ground states with $E_0 = \frac{1}{2}\hbar\omega$.

ii.) Instead, E_0, which was infinitely degenerate, splits into ~~[illegible]~~ band.

Energy eigenfcns. are $\phi_\theta(q) = \sum_{N=-\infty}^{\infty} e^{iN\theta} u_0(q - 2N\pi)$

Under symmetry, $q \to q + 2\pi$, H is invariant and

$$\phi_\theta(q + 2\pi) = e^{i\theta} \phi_\theta(q).$$

iii.) Bloch's theorem: Energy eigen fcns. obey $\phi_k(q) = e^{ikq} v_k(q)$

for k = wavenumber and $v_k(q)$ is periodic.

$\phi_k(q + 2\pi) = e^{i2\pi k} \phi_k(q)$ agrees for $2\pi k = \theta$.

iv.) Energy levels are $E_k \simeq E_0 - \alpha \cos(ka)$

$$E_\theta \simeq \frac{1}{2}\hbar\omega - \underset{\substack{\uparrow \\ \text{const.}}}{\alpha} \cos\theta$$

---- Back to Θ-vacuum. Define $T|\Psi\rangle = |\Psi'\rangle$, where

$\Psi'[A_j^g(x)] = \Psi[A_j(x)]$, for $A_j^g(x)$ is $A_j(x)$ after acting with large gauge transformation.

Explicitly, $g_1(x) = U_1(x)$ from before.

So $T|n\rangle = |n+1\rangle$, T commutes with H.

Then, $|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{-in\theta} |n\rangle$ are eigenstates of T with eigenvalue $e^{i\theta}$.

Note $\langle\theta'| e^{-iHt} |\theta\rangle = \sum_{m,n} e^{im\theta'} e^{-in\theta} \langle m| e^{-iHt} |n\rangle$

Pf. of superselection.

$$= \sum_{m,n} e^{im(\theta'-\theta)} e^{i(m-n)\theta} \langle m| e^{-iHt} |n\rangle$$

$$= \sum_{m,k} e^{im(\theta'-\theta)} e^{ik\theta} \langle k| e^{-iHt} |0\rangle$$

since H is gauge invariant.

$$= 2\pi\,\delta(\theta'-\theta) \sum_k e^{ik\theta} \langle k| e^{-iHt} |0\rangle$$

So θ-vacua cannot evolve into different ones.

Can absorb $e^{ik\theta}$ by modifying Lagrangian:

$$\langle k| e^{-iHt} |0\rangle = \int [dA_\mu]\, e^{i\int d^4x\, \mathcal{L}}$$

Associate k with topological index $k = \frac{g^2}{16\pi^2} \int d^4x \,\mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$

$$\Rightarrow \Delta\mathcal{L} = \frac{\theta g^2}{16\pi^2}\, \mathrm{tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Lecture 14.
Supplement. Miscellaneous topics for instantons.

1.) The BPST construction for the $N=1$ instanton & self-duality.

Peskin notes, Lecture 10.

In 4D, Euclidean Yang-Mills instanton is the field ~~solution~~ configuration

with $\int d^4x \, \frac{g^2}{32\pi^2} F\tilde{F} = \int d^4x \, \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} = 1$

and solves Euclidean EOMs. So, try to minimize $\int d^4x \, \frac{1}{4}(F^a_{\mu\nu})^2$

and impose constraint $\int d^4x \, \frac{g^2}{8\pi^2} F\tilde{F} = 1$

Belavin, Polyakov, Schwartz, Tyupkin PL 59B, 85 (1975)

Start with
$$
\begin{aligned}
0 &\le \int d^4x \left(F^a_{\mu\nu} \pm \tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma} F^a_{\lambda\sigma}\right)^2 \\
&= \int d^4x \left(F^a_{\mu\nu} \pm 2\,\tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma} + \tfrac{1}{4}\epsilon_{\mu\nu\lambda\sigma}\epsilon_{\mu\nu\eta\theta} F^a_{\lambda\sigma} F^a_{\eta\theta}\right) \\
&= \int d^4x \left[2(F^a_{\mu\nu})^2 \pm 2\left(\tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma}\right)\right]
\end{aligned}
$$

So, $S[A] = \int d^4x \, \frac{1}{4}(F^a_{\mu\nu})^2 \ge \left| \int d^4x \, \frac{1}{4}(F\tilde{F}) \right| = \frac{8\pi^2}{g^2}|W|$

where W is the winding of the given configuration, and

$F_{\mu\nu} \neq 0$ only in small region.

Since we want $W=1$, $S[A] = \int d^4x \, \frac{1}{4}(F^a_{\mu\nu})^2 = \frac{8\pi^2}{g^2}$

and thus we need to solve

$$F^a_{\mu\nu} = \pm \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} F^a_{\lambda\sigma}$$
, where $+$ = self-dual fields, $-$ = anti-self-dual fields.

Reduces the solution of field equations from 2nd order to 1st order.

Taubes proved the only finite-action solutions of Euclidean Yang-Mills equations are self-dual or anti-self dual.

See Shifman, chap. 20.2 for further discussion of 't Hooft symbols. $A^a_\mu \to \frac{2}{g}\eta_{a\mu\nu}\frac{x_\nu}{x^2}$, $x \to \infty$.

2) The vacuum energy of QCD.

Calculate $e^{-\mathcal{E}VT} = \int DA\, e^{-S[A]}$ functional integral,

$S[A]$ is Euclidean action, $\mathcal{E}$ is vacuum energy density, and

$V\cdot T$ is volume of Euclidean spacetime.

We can write fcn-al integral as sum over topological sectors.

$$= 1 \underbrace{\int DA}_{W=0}\, e^{-\int \frac{1}{2}\delta A\cdot(-\partial^2)\delta A} + e^{-8\pi^2/g^2}\cdot 2\underbrace{\int DA}_{\substack{W=\pm 1,\ DA \text{ path integral} \\ \text{is over } \delta A = A^\mu - A^\mu_{\text{inst.}}}}\, e^{-\int \frac{1}{2}\delta A\cdot -D^2\delta A} + \ldots$$

$$= \text{const.}\left(1 + 2\int d^4x_0\, \frac{d\rho}{\rho^5}\, C\, \frac{1}{g^8}\cdot e^{-\frac{8\pi^2}{g^2(\rho)}} + \underbrace{\ldots}_{\geq 2 \text{ inst.}}\right)$$

Remarks:

In computing fluctuations around instanton, there are 8 zero modes.

These correspond to 8 transformations that ~~do~~ ~~the~~ leave the classical solution invariant: 4 translations, 1 dilatation (scale parameter ρ), and 3 $SU(2)$ rotations.

The $\frac{1}{\rho^5}$ factor comes from dimensional analysis.

Further instanton contributions come ~~to~~ with additional $e^{-\frac{8\pi^2}{g^2(\rho)}}$ factors.

The $\left(\frac{1}{g}\right)^8$ factor comes from Jacobian for change of variables from integral over zero modes.

$C = 2^{10}\pi^6 e^{7.0539\ldots}$ from 't Hooft, Phys. Rev. D14, 3432 (1976)

Errata. D18, 2199 (1978)

Phys. Repts. 142, 357 (1986)

Treating the integration over ρ formally, as ρ becomes large, $g^2(\rho)$ grows and the semiclassical approximation breaks down.

Generalizing, contributions with n_+ instantons and n_- anti-instantons

is $\frac{1}{n_+!\, n_-!}\left[\int d^4x_0 \frac{d\rho}{\rho^5} C \frac{1}{g^8} e^{-8\pi^2/g^2(\rho)}\right]^{n_+ + n_-}$

assuming dilute gas approx. [instantons are well-separated]

Sum over all n_+ and n_- ~~....~~; with $n_+ - n_- = 1$,

$$e^{-\mathcal{E}VT} = \text{const.}\left(1 + 2\sum_{\substack{n_+-n_-=1,\\ n_+,n_-}} \frac{1}{n_+!\, n_-!}\left(\int d^4x_0 \frac{d\rho}{\rho^5} C \frac{1}{g^8} e^{-\frac{8\pi^2}{g^2(\rho)}}\right]^{n_+ + n_-} + \dots\right)$$

$$= \exp\left(2\int d^4x_0 \frac{d\rho}{\rho^5} C \frac{1}{g^8} e^{-8\pi^2/g^2(\rho)}\right)$$

Identity $e^{2x} = \sum_{n_1}\sum_{n_2} \frac{1}{n_1!}\frac{1}{n_2!} e^{n_1 x} e^{n_2 x}$

$$\Rightarrow \mathcal{E} = -2\int \frac{d\rho}{\rho^5} C \frac{1}{g^8} e^{-8\pi^2/g^2(\rho)}$$

For $\Delta\mathcal{L} = \frac{-i\theta}{32\pi^2}(F\tilde{F})$, the computation gets an additional factor $e^{i\theta(n_+ - n_-)}$

$$\Rightarrow e^{-\mathcal{E}VT} = \left[\sum_{n_+,n_-} \int d^4x \frac{d\rho}{\rho^5} C \frac{1}{g^8} e^{-8\pi^2/g^2(\rho)}\right]^{n_+ + n_-} \frac{1}{n_+!\, n_-!} e^{i\theta(n_+ - n_-)}$$

$$= \exp\left((e^{i\theta} + e^{-i\theta}) VT \int \frac{d\rho}{\rho^5} C \frac{1}{g^8} e^{-8\pi^2/g^2}\right]$$

$$\mathcal{E} = -2\cos\theta \int \frac{d\rho}{\rho^5} C \frac{1}{g^8} e^{-8\pi^2/g^2(\rho)}$$

So the energy of the QCD vacuum depends on θ: this is the underlying reason why the axion solves the θ problem of QCD; the axion couples to θ and minimizes the overall energy of the vacuum, which adjusts θ to zero.