Lecture 13.

8.2.19.

Continue studying kink in 1+1 QFT:

cf. E. Weinberg.

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)\partial_\mu \phi) - V(\phi)$$

$$V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\lambda}{4}v^4 \quad , \quad v = \sqrt{\frac{m^2}{\lambda}}$$

Define topological current

$$J^\mu_{top} = \frac{1}{2v}\epsilon^{\mu\nu}\partial_\nu\phi \quad \Rightarrow \text{conserved since } \partial_\mu J^\mu_{top} = 0$$

$$Q_{top} = \int_{-\infty}^{\infty} dx\, J^0 = \frac{1}{2v}\int_{-\infty}^{\infty} dx\, \partial_1\phi$$

$$= \frac{1}{2v}\phi(x)\Big|_{x=-\infty}^{\infty} = \begin{matrix} 1 & \text{for kink} \\ -1 & \text{for anti-kink} \end{matrix}$$

No finite energy process can change asymptotic values of field.

$\Rightarrow Q_{top}$ is conserved. Also, topological charge not manifest from symmetry of $\mathcal{L}$.

Generalize: arbitrary $V(\phi)$.

Suppose $\phi(\pm\infty)$ in different minima of V.

$$0 = \frac{d^2\phi}{dx^2} - \frac{dV}{d\phi} \quad \Rightarrow \text{Mechanical analogue: } m\ddot{x} = -\frac{\partial V}{\partial x}$$

$$\frac{d\phi}{dx} = \pm\sqrt{2V(\phi)}$$

x is "time" and ϕ is coordinate of unit mass point particle, then Newton's second law for particle motion in potential $-V(\phi)$. Solution $\phi(x)$ is motion of particle.

Virial identity:

$$\int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2\right] = \int_{-\infty}^{\infty} dx\, V(\phi)$$

relates gradient + potential terms in $\mathcal{L}$.

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Another example: ②

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\,\partial^\mu\phi) + \frac{m^4}{\lambda}\left(\cos\left(\frac{\sqrt{\lambda}}{m}\phi\right) - 1\right)$$

sine-Gordon theory.

Expand: $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - \frac{\lambda^2}{m^2}\frac{\phi^6}{6!} + \ldots$

For $\lambda \to 0$, free Klein-Gordon.

Same method: E-L equation

$$0 = \frac{d^2\phi}{dt^2} - \frac{d^2\phi}{dx^2} + \frac{m^3}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}}{m}\phi\right)$$

Periodic potential: Minimum for $\phi = \frac{2\pi m}{\sqrt{\lambda}}N \equiv Nv$, $N \in \mathbb{Z}$

Solve: $\phi(x) = Nv + \frac{2v}{\pi}\tan^{-1}\left(e^{m(x-x_0)}\right)$

$\rightsquigarrow M_{cl} = \frac{8m^3}{\lambda}$.

Ref. Coleman, Phys. Rev. D 11 2088 (1975)

Aside: sine-Gordon is dual to massive-Thirring model with fermion in 1+1

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - M)\Psi - \frac{g}{2}(\bar{\Psi}\gamma^\mu\Psi)^2$$

kink – Ψ

anti-kink – $\bar{\Psi}$

elementary ϕ — $\bar{\Psi}\Psi$ bound state

topological charge – fermion #.

Remark: Using virial theorem, can prove there are no static solutions for $D \geq 3$ in field theories with only scalar fields. cf. Derrick (1964), Hobart (1963).

Pf. Extremum condition for energy functional

$$W[\phi] \equiv \int d^D x \left[\frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi + U(\phi(x))\right]$$

$$\equiv V_1[\phi] + V_2[\phi]$$

W, V_1, V_2 are all non-negative.

Let $\phi_1(x)$ be static solution. Consider $\phi_\lambda = \phi_1(\lambda x)$

$$W[\phi_\lambda] = V_1(\phi_\lambda) + V_2(\phi_\lambda)$$
$$= \lambda^{2-D} V_1(\phi_1) + \lambda^{-D} V_2(\phi_1)$$

So $\frac{d}{d\lambda} W[\phi_\lambda] = 0$ because extremum at $\lambda = 1$

$$\Rightarrow \quad (2-D) V_1(\phi_1) = D\, V_2(\phi_1)$$

Cannot be satisfied for $D \geq 3$ unless $V_1(\phi_1) = V_2(\phi_1) = 0$.

So $\phi_1(x)$ must be space ~~independent~~ independent + equal to zero of $U(\phi)$.

Given scalar field theory in $D=4$ does not have soliton solutions, how/why do we want to study topological solitons in more realistic QFTs?

Recall non-linear EOMs corresponded to possible field configurations that were identical to vacuum solution at ~~[illegible]~~ $\pm\infty$ but topologically nontrivial + could not decay to vacuum solution.

Will now focus on non-Abelian gauge theory in 4D.

To start, recall that we motivated the study of the wrong sign potential as a result of extremizing the action via the Euler-Lagrange equations.

Equivalently, can consider Euclidean formulation of theory.

$$\mathcal{L} = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 - V(x) \quad , \quad m=1.$$

Let particle be at x_i @ $t = \frac{-t_0}{2}$ + x_f @ $t = +\frac{t_0}{2}$.

Then amplitude is evaluated by

$$\langle x_f | e^{-iHt_0} | x_i \rangle = N \int [Dx]\, e^{iS[x(t)]}$$

in path integral formulation, with $x(t)$ subject to B.Cs.

$$+ \quad S = \int_{-t_0/2}^{t_0/2} dt \; \mathcal{L}(x, \dot{x})$$

Instead of states with def. position, use energy:

$$\sum_n \langle x_f | e^{iHt_0} | n \rangle \langle n | x_i \rangle = \sum_n e^{-iE_n t_0} \langle x_f | n \rangle \langle n | x_i \rangle$$

To extract ground state, convenient to take $t \to -i\tau$ + $\tau_0 \to \infty$, then only one term survives.

Corresponding Euclidean action,

$$S_E = \int_{-\tau/2}^{\tau/2} d\tau \left[\frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right]$$

Quantizing small perturbations around a vacuum solution by extremizing the action leads to oscillatory behavior near equilibrium soln.

For multiple degenerate vacua, localizing particle in one vacuum is not protected from tunneling to other vacuum.

For double well, if particle is in left well at $t=0$, it nevertheless develops non-zero amplitude to be in right well for $t>0$.

True ground state is even superposition of two wavefcns.

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Analyze Yang-Mills theory:

Central question: what is the vacuum of Yang-Mills?

Will find that the YM Lagrangian represents a continuum of theories labeled by the parameter Θ, just as each value of the coupling constant describes a different theory.

cf. Cheng + Li, Chap. 16

Start with non-Abelian Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}, \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

Then, in analogy with non-linear solns. in scalar theories, the $g f^{abc} A^b_\mu A^c_\nu$ term provides a non-linear structure.

In scalar scenario, we studied Euclidean action.

Send Minkowski metric to Euclidean.

cf. Weinberg, Classical Solutions

Basics of YM theory: $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$

or, equivalently, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

with $A_\mu = A^a_\mu T^a$

T^a are Hermitian generators, normalized s.t. $\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$

Structure const. are totally antisymmetric: $[t^a, t^b] = i f^{abc} t^c$

Scalars & fermions are matter fields:

$$D_\mu \phi = \partial_\mu \phi - ig A_\mu \phi \Rightarrow (D_\mu\phi)_j = \partial_\mu \phi_j - ig A^a_\mu (t^a)_{jk}\phi_k$$

Under gauge transformation, $U(x)$,

$$A_\mu \to U A_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} = U A_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1}$$

$$F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$$

Note $\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} = -\frac{1}{2}\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}]$ since $-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} = -\frac{1}{2} F^a_{\mu\nu} F^{\mu\nu b} \frac{1}{2}\delta^{ab} = -\frac{1}{2}\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}]$.

We can rescale $A_\mu \to \frac{A_\mu}{g}$, then

$$F^a_{\mu\nu} \to \frac{1}{g}\partial_\mu A^a_\nu - \frac{1}{g}\partial_\nu A^a_\mu + \frac{f^{abc}}{g} A^b_\mu A^c_\nu \equiv \frac{F^a_{\mu\nu}}{g}$$

and $\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{\mu\nu a}$ with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$

Now, proceeding as before, with same logic as 1+1 kink solution, we want field configurations that asymptotically are vacuum solutions but can still carry finite, localized energy density.

What is the requirement for vacuum solution?

Classically, $F_{\mu\nu} = 0$, so $A_\mu = 0$ is an easy solution.

But, from recalling gauge transformation/gauge redundancy, it is sufficient that $A_\mu = 0 \to A_\mu = \frac{i}{g} U \partial_\mu U^{-1}$.

Hence, if A_μ asymptotes to $\frac{i}{g} U \partial_\mu U^{-1}$ at spatial infinity, then it is gauge-equivalent to zero and the gauge invariant $\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}^a F^{\mu\nu a}$ also vanishes, which is also characteristic of classical stationary solutions.

Such configurations, $A_\mu = \frac{i}{g} U \partial_\mu U^{-1}$, are called <u>pure gauge</u>.

In order to discuss gauge fields in a well-defined manner, we have to fix a gauge. The first gauge to consider is:

<u>$A_0 = 0$ gauge:</u>

Specialize to $SU(N)$.

Then $A_\mu = A_\mu^a \frac{\sigma^a}{2} = A_\mu^a \tau^a$, σ^a are Pauli matrices.

At a given moment in time, the pure gauge requirement is applied to $U(\vec{x}) \to 1$, $|\vec{x}| \to \infty$ (or any other const. matrix ind. of dir.)

So for given moment in time, work with $U(\vec{x})$ with all points at infinity identified, so $\mathbb{R}^3 \Rightarrow S^3$.

Note any $SU(2)$ matrix can be written as

$$M = A + i\vec{B} \cdot \vec{\tau} \,, \quad M \in SU(2); \quad A, \vec{B} \text{ are four real params.}$$

$$M^+ M = 1 \;\; + \;\; \det M = 1 \;\; \text{imply} \;\; A^2 + \vec{B}^2 = 1$$

So $SU(2)$ group space is topologically S^3 as well.

Can classify mappings $U(\vec{x})$ according to homotopy class, $\pi_3(S^3) = \mathbb{Z}$, so $U(\vec{x})$ are characterized by <u>winding number</u>. As a consequence, in $A_0 = 0$ gauge, there exist ∞ degenerate classical vacua (Shifman "pre-vacua") from the nontrivial topology of $SU(N)$ mappings from coordinate space to group space.

Note $U(1)$ has $\pi_3(S^1) = 0$