Lecture 12. (1)

2-1-19.

Solitons + classical field theory.

Refs. E. Weinberg - Classical Solutions in QFT

Shifman - Advanced QFT

R. Rajaraman - Solitons + Instantons

Tong - 2005 TASI Lectures on Solitons, hep-th/0509216

Coleman - The Uses of Instantons

Novikov, Shifman, Vainshtein, Zakharov - ABC of Instantons

Shifman, Vainshtein - Instantons vs. SUSY: 15 Years Later

Overview:

Analyze in classical field theory first

For usual QFT, focus first on free field theory.

Wavelike excitations of fields are quantized + interpreted as particles (impose commutation + anti-commutation relations on field + conjugate momenta)

But there are generally other solutions to classical field theory, generally when vacuum is non-trivial (connected to non-linear EOM.)

These classical solutions carry topological charge N.

Solutions are stable since they carry topological charge + cannot decay to vacuum ~~that~~ which is topologically trivial. Smooth deformations of field (pert. theory particles) around vacuum cannot change topology.

Such a $N=1$ solution with minimal energy field configuration is classically stable, cannot decay to topologically trivial field. Energy density is smooth + concentrated in finite region of space ⇒ "topological soliton."

Length scale + energy of soliton depend on consts. in Lagrangian. + field equations.

Energy of soliton is rest mass, remains finite as $\hbar \to 0$.

Examples:
- In 1+1, particle-like soliton (localized)
- In 2+1, becomes planar solution = domain wall
- In ~~1+1~~ 2+1, a particle-like soliton becomes line solution / string in 3+1.

Consider particles in 4D (not (3,1).)

They are solutions in Euclideanized spacetime = instantons.

Recall wavefunctions extend to classically forbidden regions with potential E > total E.

Requires a negative kinetic energy = evolution in Euclidean spacetime with imaginary time.

So instanton solutions are tunneling actions in Minkowski

EW sphalerons are another example. Unstable, classical solutions.

Consider simplest example of soliton in field theory:

Kink in 1+1 field theory.

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - V(\phi)$$

$$V(\phi) = \frac{-1}{2} m^2 \phi^2 + \frac{\lambda}{4}\phi^4 + \frac{\lambda}{4} v^4 = \frac{\lambda}{4}(\phi^2 - v^2)^2 \text{ with } m^2, \lambda > 0,$$

$$v = \sqrt{\frac{m^2}{\lambda}}$$

Note $\dim[\phi]=0$, $\dim[m]=1$, $\dim[\lambda]=2$.

Euler-Lagrange equation:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)-\frac{\partial\mathcal{L}}{\partial\phi}=0$$

From $\mathcal{L}=\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+\frac{1}{2}m^2\phi^2-\frac{\lambda}{4}\phi^4-\frac{\lambda}{4}v^4$

$$\Rightarrow\quad \partial_\mu\partial^\mu\phi = m^2\phi-\lambda\phi^3 = \lambda\phi\left(\frac{m^2}{\lambda}-\phi^2\right)=-\lambda\phi\left(\phi^2-v^2\right)$$

$$\Rightarrow\quad \frac{d^2\phi}{dt^2}-\frac{d^2\phi}{dx^2}=-\lambda\phi\left(\phi^2-v^2\right)$$

For static solutions, $0=\frac{d^2\phi}{dx^2}-\lambda\phi\left(\phi^2-v^2\right)$

Have $\phi(x)=0,\pm v$ as trivial solutions ($\phi(x)=0$ is unstable)

What about non-trivial solutions?

Potential energy:

$$U[\phi(x)]=\int dx\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2+V(\phi)\right]$$

$\phi(x)$ must be stationary point in general.

Consider $\phi(\infty)=v$, $\phi(-\infty)=-v$.

Lowering energy via smooth variation must respect boundary conditions (rigorous if space of field configurations is compact.)

Cannot be either vacuum solution.

Rescale: $\phi=vf$, $u=mx$.

$$0=\frac{d^2f}{du^2}-f(f^2-1)$$

$$E=\frac{m^3}{\lambda}\int du\left[\frac{1}{2}\left(\frac{df}{du}\right)^2+\frac{1}{4}(f^2-1)^2\right]$$

Note non-perturbative

Spatial variation is $\mathcal{O}(1)$ in units of u & $\mathcal{O}(m^{-1})$ in units of x.

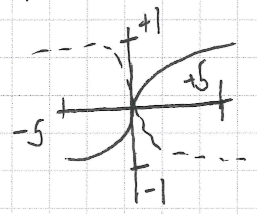
Solve E-L equation.

$$0 = \frac{d}{dx}\left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - \frac{\lambda}{4}\left(\phi^2-v^2\right)^2\right]$$

So brackets is independent of x. For $\phi(\infty)$, brackets vanishes,

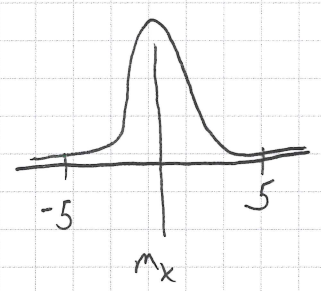
so $$\frac{d\phi}{dx} = \pm\sqrt{\frac{\lambda}{2}}\left(\phi^2-v^2\right) \quad + \quad \phi(x) = \pm v \tanh\left(\frac{m}{\sqrt{2}}(x-x_0)\right)$$

$+$ kink solution
$-$ anti-kink solution



Energy density:

$$\epsilon(x) = \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi)$$
$$= \frac{m^4}{2\lambda}\,\mathrm{sech}^4\left[\frac{m}{\sqrt{2}}(x-x_0)\right]$$



$$M_{cl} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} = \int_{-\infty}^{\infty} dx\ \epsilon(x)$$

Outside x_0, field indistinguishable from vacuum.
Localized energy profile.

Moving kink: $\phi(x,t) = v\tanh\left[\frac{m}{\sqrt{2}}\left(\frac{x-ut-x_0}{\sqrt{1-u^2}}\right)\right]$; $E = \frac{M_{cl}}{\sqrt{1-u^2}}$

Key: In contrast to normal QFT, we think of a classical ground state implying a quantum ground state + neglect possible effects from possible degenerate, classical vacua that can arise from nonlinear/interaction terms in the Lagrangian.

$V(\phi)$ had multiple degenerate vacua + field approached distinct vacua at spatial infinity. Topology of vacua gives rise to topological solitons.

Cf. Introduction to topology in Lecture 9.