Lecture 11.

1-25-19

$\pi^0 \to \gamma\gamma$ decay calculation + 't Hooft anomaly matching.

From last time, chiral Lagrangian arises from operating on chiral symmetry breaking vacuum with Goldstone multiplet, requiring invariance under original $SU(2)_L \times SU(2)_R$ symmetry (for $N_f = 2$).

As usual, low energy composite degrees of freedom transform as irreps. of residual gauge symmetry $(U(1)_{em})$.

They also fall into irreps. of approximate global symmetry,

$SU(2)_{isospin}$ = diagonal subgroup of $SU(2)_L \times SU(2)_R$.

In particular, proton + neutron are doublet of $SU(2)_{isospin}$.

$\psi = \begin{pmatrix} p \\ n \end{pmatrix}$. [Recall, Gell-Mann Eightfold way is same analysis using $SU(3)_{flavor}$ symmetry.]

For $N_f = 2$ QCD, the Goldstone multiplet is a triplet of $SU(2)_{isospin}$, since they correspond to the coset space of $SU(2)_L \times SU(2)_R / SU(2)_V$. $U \to g_L U g_R^\dagger$, with $g_L \in SU(2)_L$, $g_R \in SU(2)_R$

$$U = \exp\left(2i \frac{\pi^a \tau^a}{f_\pi}\right) = \exp\left(\frac{i}{f_\pi}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0 \end{pmatrix}\right)$$

$$\pi^0 = \pi^3 \quad , \quad \pi^\pm = \frac{1}{\sqrt{2}}\left(\pi^1 \pm i\pi^2\right)$$

Additional matter fields, like proton + neutron, have tree-level masses: $\psi = \begin{pmatrix} p \\ n \end{pmatrix} = \psi_L + \psi_R$

$m_N \bar{\psi}\psi = m_N \bar{\psi}_L \psi_R + m_N \bar{\psi}_R \psi_L$ is invariant for $g_L = g_R$.

Leading term from chiral Lagrangian and nucleon mass & $SU(2)_L \times SU(2)_R$ invariant interaction term is

$$\mathcal{L} = \frac{f_\pi^2}{4} \, tr\left[(\partial_\mu U)(\partial^\mu U)^\dagger\right] + \bar{\Psi}_L i\partial\!\!\!/ \Psi_L + \bar{\Psi}_R i\partial\!\!\!/ \Psi_R - m_N\left(\bar{\Psi}_L U \Psi_R + \bar{\Psi}_R U^\dagger \Psi_L\right)$$

$$= \left(+\frac{1}{2}(\partial_\mu \pi^a)(\partial^\mu \pi^a) + \dots\right) + \bar{\Psi}(i\partial\!\!\!/ - m_N)\Psi + i\frac{2m_N}{f_\pi}\pi^a\left(\bar{\Psi}\gamma^5\tau^a\Psi + \dots\right)$$

$\tau^a = \frac{\sigma^a}{2}$, $\tau^3 = \begin{pmatrix} \frac{1}{2} & \\ & -\frac{1}{2}\end{pmatrix}$, dictates interaction between π^0 & protons: $i\frac{m_N}{f_\pi}\pi^0(\bar{p}\gamma^5 p)$. Axial scalar coupling.

Consider generic calculation: scalar decay to $\gamma\gamma$ via axial coupling to charged fermions.

Toy Lagrangian:

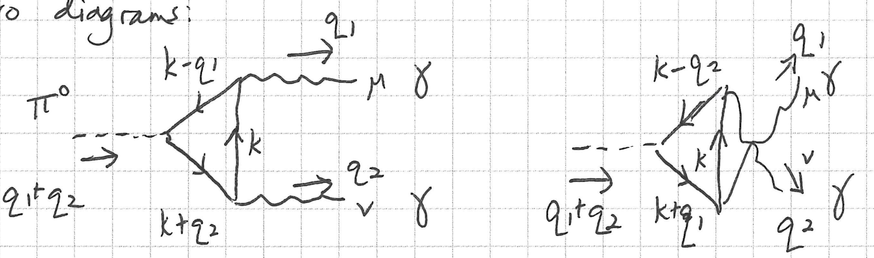
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu \pi_0)(\partial^\mu \pi^0) - \frac{1}{2}m_\pi^2 \pi_0 \pi^0 + \bar{\Psi}(iD\!\!\!\!/ - m)\Psi + i\lambda \pi_0 \bar{\Psi}\gamma^5\Psi$$

$iD\!\!\!\!/ = i\partial\!\!\!/ - eA\!\!\!/$, unit charge fermion.

Calculation is very similar to $h \to gg$ or $h \to \gamma\gamma$.

Two diagrams:

Ref. Schwartz Chap. 30.



first diagram

$$i\mathcal{M}_1 = (-1)\int \frac{d^4k}{(2\pi)^4} \, Tr\left[(-ie\gamma^\mu)\frac{i(k\!\!\!/ + m)}{k^2 - m^2}(-ie\gamma^\nu)\frac{i(k\!\!\!/ + q\!\!\!/_2 + m)}{(k+q_2)^2 - m^2}(-\lambda\gamma^5)\frac{i(k\!\!\!/ - q\!\!\!/_1 + m)}{(k-q_1)^2 - m^2}\right]\epsilon_\mu^*\epsilon_\nu^*$$

$\mathcal{M}_2$ is same as $\mathcal{M}_1$ with $\mu \leftrightarrow \nu$ and $1 \leftrightarrow 2$.

Remark: ① By counting γ matrices in trace numerator, can conclude $\mathcal{M}_1$ is <u>finite</u>. So, no need for 't Hooft-Veltman prescription for γ^5 in dim. reg.

[Details: Leading linear divergence is $[\gamma^\mu \not{k} \gamma^\nu \not{k} \gamma^5 \not{k}] \sim 0$, 5 γ matrices. vanishes.]

[Also, log divergence with two powers of loop momentum is symmetric in $\not{k}$, will vanish since

$$\mathrm{Tr}[\alpha\beta\gamma\delta 5] = -4i\,\epsilon^{\alpha\beta\gamma\delta} \quad]$$

② Also, first nonzero trace with γ^5 requires 4 γ matrices.

$$\text{So, } i\mathcal{M}_1 = i\lambda e^2 \epsilon^*_\mu \epsilon^*_\nu \int \frac{d^4k}{(2\pi)^4} \cdot \frac{1}{k^2-m^2} \frac{1}{(k+q_2)^2-m^2} \frac{1}{(k-q_1)^2-m^2}$$

$$\mathrm{Tr}[\gamma^\mu (\not{k}+m) \gamma^\nu (\not{k}+\not{q}_2+m) \gamma^5 (\not{k}-\not{q}_1+m)]$$

Evaluate trace. Only keep terms with 4 γ matrices.

$$\begin{aligned}
\mathrm{Tr} &= \mathrm{Tr}[\gamma^\mu (\not{k}) \gamma^\nu (\not{k}+\not{q}_2) \gamma^5 m] \\
&+ \mathrm{Tr}[\gamma^\mu \not{k} \gamma^\nu m \gamma^5 (\not{k}-\not{q}_1)] \\
&+ \mathrm{Tr}[\gamma^\mu m \gamma^\nu (\not{k}+\not{q}_2) \gamma^5 (\not{k}-\not{q}_1)] \\
&= \mathrm{Tr}[\gamma^\mu \not{k} \gamma^\nu \not{q}_2 \gamma^5] m + m\,\mathrm{Tr}[\gamma^\mu \not{k} \gamma^\nu \gamma^5 (-\not{q}_1)] \\
&+ m\,\mathrm{Tr}[\gamma^\mu \gamma^\nu \not{k} \gamma^5 (-\not{q}_1)] + m\,\mathrm{Tr}[\gamma^\mu \gamma^\nu \not{q}_2 \gamma^5 \not{k}] \\
&- m\,\mathrm{Tr}[\gamma^\mu \gamma^\nu \not{q}_2 \gamma^5 \not{q}_1]
\end{aligned}$$

Note $2^{\text{nd}} + 3^{\text{rd}}$ terms are

$$m\,\mathrm{Tr}[\gamma^\mu (\not{k}\gamma^\nu + \gamma^\nu \not{k}) \gamma^5 (-\not{q}_1)] = m\,\mathrm{Tr}[\gamma^\mu (2k^\nu) \gamma^5 (-\not{q}_1)] = 0$$

Also, $1^{\text{st}} + 4^{\text{th}}$ terms ~~are~~ also sum to zero.

$$\text{Get } \mathrm{Tr} = +m\,\mathrm{Tr}[\gamma^\mu \gamma^\nu \not{q}_2 \not{q}_1 \gamma^5]$$

$$= -4im\,\epsilon^{\mu\nu\alpha\beta} q_{2\alpha} q_{1\beta}.$$

Combine denominators using Feynman params.

$$\frac{1}{D_1 D_2 D_3} = \int_0^1 dx \int_0^{1-x} dy \; \frac{2}{\left(k^2 - m^2 + 2k\cdot q_2 x + x q_2^2 - 2yk\cdot q_1 + y q_1^2\right)^3}$$

Note $q_1^2 = q_2^2 = 0$ (massless photons).

$$\ell = k + xq_2 - yq_1$$

$$\ell^2 = (k + xq_2 - yq_1)^2$$

$$\Delta = m^2 + (xq_2 - yq_1)^2$$

$$\frac{1}{D_1 D_2 D_3} = \int_0^1 dx \int_0^{1-x} dy \; \frac{2}{(\ell^2 - \Delta)^3}$$

$$\Delta = m^2 - 2xy\, q_1\cdot q_2 = m^2 - xy\,(2q_1\cdot q_2)$$

Note $(q_1+q_2)^2 = 2q_1\cdot q_2 = m_\pi^2$.

For $m >> m_\pi^2$, neglect m_π^2. No other x or y dependence in integral:

$$\int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \int_0^{1-x} dy \; \frac{2}{(\ell^2 - \Delta)^3}$$

$$= \iint dx\,dy\; 2 \cdot \frac{(-1)^3 i}{(4\pi)^2} \cdot \frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)} \left(\frac{1}{\Delta}\right)^{3-\frac{d}{2}}$$

$$= \iint dx\,dy\; \frac{-i}{16\pi^2} \cdot \frac{1}{\Delta}$$

$$= \frac{-i}{16\pi^2} \cdot \frac{1}{m^2} \cdot \frac{1}{2}$$

We get
$$i\mathcal{M}_1 = i\lambda e^2 \epsilon^*_\mu \epsilon^*_\nu \cdot \left(-4im\, \epsilon^{\mu\nu\alpha\beta} q_{2\alpha} q_{1\beta}\right) \cdot \left(\frac{-i}{16\pi^2}\right) \frac{1}{2m^2}$$
$$= \frac{-i}{8\pi^2} \frac{\lambda e^2}{m} \cdot \epsilon^*_\mu \epsilon^*_\nu \, \epsilon^{\mu\nu\alpha\beta} q_{2\alpha} q_{1\beta}$$

Note $\mathcal{M}_2 = \mathcal{M}_1$ since $\mu \leftrightarrow \nu$ + $1 \leftrightarrow 2$ gives same ϵ contraction.

So,
$$i\mathcal{M}_{tot} = \frac{-i}{4\pi^2} \frac{\lambda e^2}{m} \epsilon^*_\mu \epsilon^*_\nu \, \epsilon^{\mu\nu\alpha\beta} q_{2\alpha} q_{1\beta}$$

$$\mathcal{M}_{tot} = \frac{1}{4\pi^2} \frac{\lambda e^2}{m} \epsilon^{\mu\nu\alpha\beta} \epsilon^*_\mu \epsilon^*_\nu \, q_{1\alpha} q_{2\beta}$$

Corresponding decay width, identical final states

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{1}{2m_\pi} \cdot \frac{1}{2} \left(\int \frac{d\Omega_{cm}}{4\pi} \, \frac{1}{8\pi} \cdot \frac{2|\vec{p}_1|}{E_{cM}} \right) |\mathcal{M}|^2$$

via toy Lagrangian

Use $|\vec{p}_1| = E_1 = \frac{m_\pi}{2}$ + $E_{cM} = m_\pi$.

$$\Gamma = \frac{1}{2m_\pi} \cdot \frac{1}{2} \cdot \frac{1}{8\pi} \cdot \frac{\lambda^2 e^4}{16\pi^4 m^2} \left(\epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta} \right) \left(\epsilon^{\rho\sigma\gamma\delta} q_{1\gamma} q_{2\delta} \right)$$

Use eqn. A.30 from P+S:

$$\epsilon^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^\mu{}_\rho \delta^\nu{}_\sigma - \delta^\mu{}_\sigma \delta^\nu{}_\rho \right)$$

$$\Gamma = \frac{1}{512\pi^5} \frac{\lambda^2 e^4}{m_\pi m^2} (+2) \left(q_1 \cdot q_2 \right)^2$$

Use $q_1 \cdot q_2 = \frac{m_\pi^2}{2}$,

$$\Gamma = \frac{1}{1024\pi^5} \frac{\lambda^2 e^4 m_\pi^3}{m^2} = \frac{\lambda^2 \alpha_e^2}{64\pi^3} \frac{m_\pi^3}{m^2}$$

Using the chiral Lagrangian, we identify $m = m_N \sim 939$ MeV + $\lambda = \frac{m_N}{f_\pi}$, so

$$\Gamma = \frac{\alpha_e^2}{64\pi^3} \frac{m_\pi^3}{f_\pi^2} \simeq 7.77 \text{ eV},$$ compared to exp. 7.73 ± 0.16 eV.

(Schwartz, p. 620)

This calculation relied on an explicit matrix element + $\mathcal{L}$ using composite degrees of freedom generated by chiral Lagrangian.

However, an alternate way to calculate $\pi^0 \to \gamma\gamma$ is to recognize the triangle diagram as the same as the anomaly calculation for $SU(2)_A$ w.r.t. EM currents.

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Recall, from the discussion of Goldstone's theorem, that currents operating on the vacuum excite massless Goldstone modes:

$$\langle \pi(\vec{q}) | j_\mu(y) | \Omega \rangle = i q_\mu F e^{i\vec{q}\cdot\vec{y}}$$

Recall also that the π-multiplet corresponds to the $SU(2)_L \times SU(2)_R / SU(2)_{isospin} = SU(2)_{\substack{axial\\isospin}}$ symmetry group.

$$SU(2)_{\substack{axial\\isospin}} \Rightarrow \bar{q}\gamma^\mu \gamma^5 \tau^a q = j^{\mu 5 a}$$

Ref. P&S, section 19.3

So, identifying $a=3$ as neutral pion, $\pi^0 \to \gamma\gamma$ can be calculated from the non-zero anomaly between $j^{\mu 5 a}$ and $F_{\mu\nu} F^{\mu\nu}$:

$$\partial_\mu j^{\mu 5 a} = -\frac{e^2}{16\pi^2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} \, \mathrm{tr}[\tau^a Q^2]$$

Interpretation: $j^{\mu 5 3}$ annihilates a π^0 meson, axial vector anomaly contributes to $\pi^0 \to 2\gamma$ decay.

$$\mathrm{tr}[\tau^a Q^2] = \frac{1}{2}\left(\left(\frac{2}{3}\right)^2 - \left(\frac{-1}{3}\right)^2\right) = \frac{1}{6}.$$ From $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$ isospin doublet.

Using $\langle q_1, q_2 | j^{\mu 5 3}(q_1+q_2) | \Omega \rangle = \epsilon^*_\mu \epsilon^*_\nu \mathcal{M}^{\alpha\mu\nu}(q_1, q_2)$

we identify $i q_\alpha \mathcal{M}^{\alpha\mu\nu} = -\frac{e^2}{4\pi^2} \epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta}$ as before.

Parametrically, $i\mathcal{M}(\pi^0 \to \gamma\gamma) = iA\, \epsilon^*_\mu \epsilon^*_\nu \epsilon^{\mu\nu\alpha\beta} q_{1\alpha} q_{2\beta}$

$$+ \quad A = \frac{e^2}{4\pi^2} \frac{1}{f_\pi}$$

Since $N_c = 3$,

$$\partial_\mu j^{\mu 5 a} = \frac{-e^2}{32\pi^2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}$$

Hence, the anomaly determines the matrix element structure of $\pi^\circ \to \gamma\gamma$, and provides an independent check on how many colors are in $SU(3)_c$.

The fact that $\pi^\circ \to \gamma\gamma$ can be calculated from either composite fields or elementary fields is an example of 't Hooft anomaly matching.

Ingredients:

1. Color is confined in a strongly interacting gauge theory.
2. Three currents $j^{\mu i}, j^{\nu j}, j^{\mu k}$ all have nonzero anomalies + are realized as currents of manifest symmetries.

Then there must exist massless composite (color-singlet) fermions in some rep. R of the flavor group s.t.

$$\underbrace{\mathrm{tr}_{c,f}\left[t^i\{t^j, t^k\}\right]}_{\text{elementary fermions}} = \underbrace{\mathrm{tr}_f\left[T_R^i\{T_R^j, T_R^k\}\right]}_{\text{composite fermions}}$$

where T_R^i is the rep. matrix for t^i in the rep. R.

Ref. Cargese Summer Inst. 1979
't Hooft.