Cluster of Excellence Precision Physics, Fundamental Interactions and Structure of Matter
PRISMA

Lecture 10.

Scale anomaly / Weyl anomaly / trace anomaly.

Chiral Lagrangian / Goldstone field theory, $\pi^0 \to \gamma\gamma$,

't Hooft anomaly matching.

Scale anomaly.

Refs. P+S, 19.5

Shifman, Section 36.

Consider QFT with massless fields + only dim-less couplings.

For example, Yang-Mills theory.

$$S = \int d^4x \left(\frac{-1}{4g_0^2}\right) G^a_{\mu\nu} G^{\mu\nu a}$$

g_0 = bare coupling const.

Classical symmetry: scale invariance: $x \to \lambda^{-1} x$, $A^a_\mu \to \lambda A^a_\mu$

Scale transformations generated by dilatation current

$$j^D_\nu = x^\mu \Theta_{\mu\nu}$$

↳ symmetric energy-momentum tensor

In YM,

$$\Theta_{\mu\nu} = \frac{-1}{g^2}\left(G^a_{\mu\alpha} G_\nu^{\ \alpha a} - \frac{1}{4} g_{\mu\nu} G^a_{\alpha\beta} G^{\alpha\beta a}\right)$$

Classically, j^D_ν is conserved, hence $\partial^\nu j^D_\nu = \Theta_\mu^{\ \mu} = 0$

equivalence to trace in P+S

However, at quantum level, $\Theta_\mu^{\ \mu} = 0$ is not true.

Consider dim. reg. : $d = 4 - \epsilon$.

We have $\int d^{4-\epsilon}x \; G^2_{\mu\nu}$, which is not scale invariant

but the transformation is now prop. to ϵ. Also,

$\frac{1}{g_0^2}$ in terms of renormalized coupling depends on $\frac{1}{\epsilon}$.

(Formally infinite)

In the limit of $\epsilon \to 0$, we have non-zero result.

$$\delta S = \int d^{4-\epsilon}x \left[\frac{-1}{4}\left(\frac{1}{g^2} + \frac{\beta_0}{8\pi^2}\frac{1}{\epsilon}\right)(\lambda^{\epsilon}-1)\, G^a_{\mu\nu} G^{a\mu\nu} \right]$$

$$\to \int d^4x \, \ln\lambda \left(\frac{-\beta_0}{32\pi^2} G^a_{\mu\nu} G^{a\mu\nu} \right)$$

$\beta_0 = \frac{11N}{3}$ is β-fcn. for $SU(N)$.

So, $\quad \Theta_\mu{}^\mu = -\frac{\beta_0}{32\pi^2} G^a_{\mu\nu} G^{a\mu\nu}$

Massless fermions: $\beta_0 = \frac{11}{3}N - \frac{2}{3}N_f$ for N_f fundamentals.

Famously, classically scale invariant theories generate a scale from quantum effects. Dimensional transmutation

- YM theories: IR Landau pole, confining scale
- Yukawa theories: UV Landau pole.
- $U(1)$ theories: UV Landau pole.
- ϕ^4 theories: Criticality / Coleman-Weinberg.

Chiral Lagrangian of $N_f = 2$ QCD / Goldstone field theory.

Schwartz Sect. 28

Recap: Goldstone's theorem. Every spontaneously broken cont. symm. gives rise to massless field.

P+S, 11.1

Consider conserved Noether current.

$$\partial_\mu j^\mu = 0$$

$$Q = \int d^3x \, j_0(x) = \int d^3x \sum_m \frac{\partial \mathcal{L}}{\partial \dot\phi_m} \frac{\delta \phi_m}{\delta \alpha}$$

is an operator

$$\pi_m = \frac{\delta \mathcal{L}}{\delta \dot\phi_m}, \text{ canonical conj.} \qquad [\phi_n(\vec{x}), \pi_m(\vec{y})] = i\delta^3(\vec{x}-\vec{y})$$

$$[Q, \phi_n(\vec{y})] = \sum_m \int d^3x \, [\pi_m(\vec{x}), \phi_n(\vec{y})] \cdot \frac{\delta \phi_m(\vec{x})}{\delta \alpha}$$

$$= -i \frac{\delta \phi_n(\vec{y})}{\delta \alpha}$$

Q generates symmetry transformation.

$$[H, Q] = i \partial_t Q = 0$$

Have conserved charge no matter what vacuum.

Now, Q behaves diff. in symmetric vs. broken vacua.

$$Q \, |\Omega\rangle_{sym} = 0$$

$$Q \, |\Omega\rangle_{SSB} \neq 0$$

For $H|\Omega\rangle = E_0 |\Omega\rangle$

then $HQ|\Omega\rangle = [H,Q]|\Omega\rangle + QH|\Omega\rangle = E_0 Q|\Omega\rangle$

& $Q|\Omega\rangle$ is a family of degenerate ground states.

Also key: $|\pi(\vec{p})\rangle = -\frac{2i}{F} \int d^3x \, e^{-i\vec{p}\cdot\vec{x}} \, j_0(x) |\Omega\rangle$

constructs states of 3-momentum $\vec{p}$ from vacuum.

with $E(\vec{p}) + E_0$. Mass dim. $[F] = 1$.

Since $|\pi(\vec{0})\rangle = -\frac{2i}{F} Q |\Omega\rangle$ has E_0 energy,

must have $E(\vec{p}) \to 0$ as $\vec{p} \to 0$ & states must have massless dispersion relation.

Hence, $|\pi(\vec{p})\rangle$ are Goldstone bosons.

Multiply by $\langle \pi(\vec{q})|$ & integrate over $\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{y}}$

$\Rightarrow \langle \pi(\vec{q}) | j_0(y) | \Omega\rangle = i\omega_q F e^{i\vec{q}\cdot\vec{y}}$

using $\langle \pi(\vec{q}) | \pi(\vec{p})\rangle = 2\omega_p (2\pi)^3 \delta^3(\vec{q} - \vec{p})$ normalization.

Leads to $\langle \pi(\vec{q}) | j_\mu(y) | \Omega\rangle = i q_\mu F e^{i\vec{q}\cdot\vec{y}}$

Remark: symmetries, even approximate global symmetries, are useful tools to understand dynamics. Notably, they govern RG behavior $\Rightarrow$ most couplings renormalize prop. to itself.

Study QCD, $N_f = 2$. Neglect EW, work in broken phase.

$SU(3)_c\,[\times U(1)_{em}]$ ↳ perturbation, neglect for now

$$\mathcal{L} = \bar{u}\, i\cancel{\partial}\, u + \bar{d}\, i\cancel{\partial}\, d - m_u \bar{u}u - m_d \bar{d}d - \frac{1}{4}\mathrm{Tr}\left[G_{\mu\nu}G^{\mu\nu}\right]$$

Levels of symmetry:

$m_u = m_d = 0$. $\quad U(2)_u \times U(2)_d$.

$m_u = m_d = M \quad U(2)$

$$U(2)_u \times U(2)_d = U(2)_L \times U(2)_R = SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R$$
$$= SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$$

$$\mathcal{L} = \bar{u}_L\, i\cancel{\partial}\, u_L + \bar{u}_R\, i\cancel{\partial}\, u_R + \bar{d}_L\, i\cancel{\partial}\, d_L + \bar{d}_R\, i\cancel{\partial}\, d_R + \frac{-1}{4} G^a_{\mu\nu}G^{a\mu\nu}$$

$g_L \in SU(2)_L$: $\begin{pmatrix} u^L \\ d^L \end{pmatrix} \to g_L \begin{pmatrix} u^L \\ d^L \end{pmatrix}$ + same for $L \leftrightarrow R$.

Or, $Q = \begin{pmatrix} u \\ d \end{pmatrix}$ has $Q \to e^{i(\theta_a \tau^a + \gamma^5 \beta_a \tau^a)} Q$

for θ_a, β_a as infinitesimal sym. params. of $SU(2)_V + SU(2)_A$.

Correspond Noether currents:

$$j^a_\mu = \bar{Q}\,\tau^a \gamma^\mu Q \qquad j^{5a}_\mu = \bar{Q}\,\tau^a \gamma_\mu \gamma^5 Q$$
$$j^V_\mu = \bar{Q}\,\gamma^\mu Q \qquad j^A_\mu = \bar{Q}\,\gamma_\mu \gamma^5 Q$$

Focus on j^a_μ = isospin symmetry

j^{5a}_μ = axial isospin

j^V_μ = baryon #

j^A_μ = anomalous axial

Ansatz:
$$\langle \bar{u} u \rangle = \langle \bar{d} d \rangle = \Lambda^3_{QCD}$$

Quarks form a chiral condensate that spontaneously breaks (global) chiral symmetry. Treat as domininat source of chiral symmetry breaking in QCD below Λ_{QCD}.

Spont. broken syms. give massless goldstones, breaking pattern is $SU(2)_L \times SU(2)_R \to SU(2)_V$.

$\Rightarrow$ 3 massless d.o.f. $\pi^0, \pi^{+/-}$, coset space of
$$SU(2)_L \times SU(2)_R / SU(2)_V .$$

Recall, Lagrangian still respects symmetry of $SU(2)_L \times SU(2)_R$, just have to expand around SSB vacuum.

$\Rightarrow$ Chiral Lagrangian.

Refs. Pich hep-ph/9502366
Scherer hep-ph/0210398
Donoghue, Golowich, Holstein - Dynamics of the SM
Weinberg - Vol. II, Chap. 19.
Schwartz - Sect. 28.

Follow Weinberg. 19.7.

Model QCD vacuum as N_f quarks:
$$\langle q_{La} \, q^{\dagger}_{Rb} \rangle = \tfrac{1}{2} \Delta \, \delta_{ab}$$
dim. 3

flavor, diagonal subgroup

Acting on the vacuum by $SU_L(N_f) \times SU_R(N_f)$ gives
$$\langle q_{L,a} \, q^{\dagger}_{R,b} \rangle = \tfrac{1}{2} \Delta \, U_{ab} \quad \text{for} \quad U \in SU(N_f).$$

Can represent the broken symmetry ground state by expanding about $U_{ab} = \delta_{ab}$, respecting original global symmetries $SU(N_f) \times SU(N_f)$.

Under global $SU(N_f) \times SU(N_f)$, $\quad U \to e^{i\alpha\cdot t}\, U\, e^{-i\beta\cdot t}$

generated by α + β.

Terms like $\operatorname{tr} U^2$, $\operatorname{tr} U^4$ are forbidden, not invariant.

Any term in $\mathcal{L}$ built from $U^\dagger U = \mathbb{1}$, trivial.

Must only have derivative terms:

Two derivatives: $\mathcal{L} = F^2 \operatorname{tr}[\,\partial_\mu U^\dagger \partial^\mu U\,]$

(include EW interactions, get $(D_\mu U)^\dagger (D^\mu U)$)

Expand $U(x)$ around $\mathbb{1}$: $\quad U(x) = e^{2i\pi^i(x)t^i/f_\pi}$ ↳ canonical normalization of kinetic term

Nonlinear sigma model constructed from U

= Goldstone field theory for realizing interactions of nonlinearly realized sym. structure.

Explicitly, $\quad U(x) = \exp\left(2i\dfrac{\pi^a\tau^a}{f_\pi}\right) = \exp\left(\dfrac{i}{f_\pi}\begin{pmatrix}\pi^0 & \sqrt{2}\pi^- \\ \sqrt{2}\pi^+ & -\pi^0\end{pmatrix}\right)$

$$\pi^0 = \pi^3, \qquad \pi^\pm = \frac{1}{\sqrt{2}}(\pi^1 \pm i\pi^2)$$

((compare to $W^{+/-}$, W^3 in $SU(2)$ breaking)

$$\begin{aligned}\mathcal{L} = \frac{f_\pi^2}{4} \operatorname{tr}\left[(D_\mu U)(D^\mu U)^\dagger\right] &+ L_1 \operatorname{tr}\left[(D_\mu U)(D_\mu U)^\dagger\right]^2 \\ &+ L_2 \operatorname{tr}\left[(D_\mu U)(D_\nu U)^\dagger\right] \operatorname{tr}\left[(D_\nu U)^\dagger (D_\mu U)\right] \\ &+ L_3 \operatorname{tr}\left[(D_\mu U)(D_\mu U)^\dagger (D_\nu U)(D_\nu U)^\dagger\right] + \ldots\end{aligned}$$

Consider two derivs.

$$\begin{aligned}\frac{f_\pi^2}{4} \operatorname{tr}\left[(D_\mu U)(D^\mu U)^\dagger\right] &= \frac{1}{2}(\partial_\mu \pi^0)(\partial^\mu \pi^0) + (D_\mu \pi^+)(D_\mu \pi^-)^\dagger \\ &+ \frac{1}{f_\pi^2}\left(-\frac{1}{3}\pi^0\pi^0 D_\mu\pi^+ D_\mu\pi^- + \ldots\right] + \frac{1}{f_\pi^4}\left(\ldots\right) + \ldots\end{aligned}$$

Dictates all interactions from symmetry structure,

just measure parameters f_π, L_1, L_2, …

Also, Wess-Zumino-Witten term.

Wess, Zumino - Phys. Lett. 37 B 95 (1971)

Witten - Nucl. Phys. B 223, 422 (1983)
Comm. Math. Phys. 92, 455 (1984)

$$S_{WZ} = \frac{ic}{5} \int d^5y \, \epsilon^{\mu\nu\rho\sigma\eta} \, \mathrm{tr}\left[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)(U^\dagger \partial_\sigma U)(U^\dagger \partial_\eta U) \right]$$

Total derivative.

Variation of action is purely 4D expression.

$$c = \frac{n}{48\pi^2}, \quad n \in \mathbb{Z}$$